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MATHEMATICS magazine

MATHEMATICS MAGAZINE

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Address editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to a member of the executive committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *Foreword* which states in simple terms what the paper is about.

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MATHEMATICS MAGAZINE

VOL. 29, No. 3, Jan.-Feb. 1956

CONTENTS

OUR CONTRIBUTORS

S. M. Ulam is a native of Poland, and one of the famous Polish school of mathematicians headed by Banach. After coming to this country in the thirties, Dr. Ulam became a member of the Society of Fellows of Harvard University and subsequently joined the faculty of the University of Wisconsin. For the past several years he has been a Senior Scientist at the Los Alamos Scientific Laboratory. A Mathematician whose interests are both wide and deep, he has made significant contributions to many fields, including functional analysis, group theory, measure theory, ergodic theory and mathematical physics.

Glenn James was born in Wabash, Indiana, in 1882. A graduate of the Universities of Indiana (A.B.'05, A.M.'11) and Columbia (Ph.D. '17), he taught at Michigan State, Purdue and Carnegie Tech before joining the faculty of U.C.L.A., where he taught for many years. With his son R. C. James he compiled and edited the "Mathematics Dictionary." Since 1947 he has been Managing Editor of the MATHEMATICS MAGAZINE.

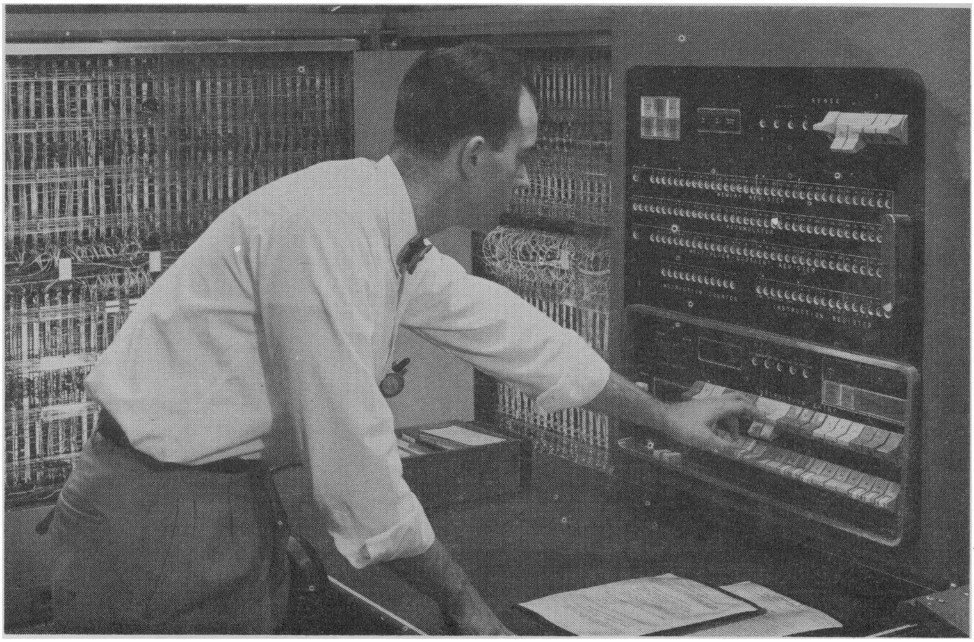
Peter Yff is a graduate of Roosevelt College (B.S.'47) and the University of Chicago (M.S.'48). During World War II he served in the U.S. Navy, and from 1951-55 he was first, Assistant Professor and then Chairman of the Mathematics Department at the American University, Beirvt, Lebanon. He has now returned for more graduate study at the University of Illinois.

Harry S. Vandiver, Distinguished Professor of Mathematics, University of Texas, was born in Philadelphia in 1882. He is well known for his extensive research in algebra and number theory and has received various honors for his work, including the Cole Prize ('31), election to the National Academy of Sciences and an honorary degree of D.Sc. from the University of Pennsylvania. Prominent in the American Mathematical Society, he was named Colloguium Lecturer ('35) and Vice President ('35-37). The present article is one of a series.

Milo W. Weaver is a graduate of the University of Texas (B.A. '35, M.A.'50). After teaching mathematics and science in the public schools of Texas for ten years, he returned to the University as Instructor of Extension Mathematics in 1945. A student of Prof. Vandiver, Mr. Weaver is interested in abstract algebra, particularly in semi-groups.

W. R. Ransom was born in Chicago in 1876. He is a graduate of Tufts College, Mass., and took graduate work at Harvard University. He taught mathematics and sometimes physics and astronomy at Tufts over a period of fifty years. Among the students in his classes later to become famous were Norbert Wiener and Vannevar Bush. He is the author of *A Hundred Mathematical Curiosities*, and *Geometry Growing* as well as textbooks.

(continued on inside of back cover)



Mathematical Analyst *Keith Kersery loads jet transport flutter problem into one of Lockheed's two 701's. On order: two 704's to help keep Lockheed in forefront of numerical analysis and production control data processing.*

ON CERTAIN SEQUENCES OF INTEGERS DEFINED BY SIEVES

Verna Gardiner; R. Lazarus,
N. Metropolis and S. Ulam

The sequence of primes can be defined by the sieve of Eratosthenes. One can think of variations in the definition of a sieve. The following problem was studied by means of actual calculation on an electronic computing machine (partly with the aim of trying to develop "coding procedures" which would obviate the need for large memories which at first appear necessary for problems of this sort). Consider the sequence of all positive integers, 1, 2, 3, ... We shall now strike out from this sequence every *second* term by counting from 1. The odd integers will be left. We shall now strike out every *third* integer in the remaining sequence, again starting to count from 1, but considering only the remaining integers. We shall obtain a second sequence of integers. The next step is to strike out every *fourth* integer counting only the remaining ones and we obtain another subsequence. We can continue this process indefinitely. It is obvious that infinitely many integers will remain after we have completed the process. This sieve is different from that of Eratosthenes since in striking out all multiples of successive integers we count off only among the remaining ones. It could perhaps be called a sieve of Josephus Flavius. The result of this sieve is a sequence of integers of a density much smaller than that of the primes.

Another sieve could be the following: consider again the sequence of integers starting with 1. We shall strike out from it every second term. Apart from 1, the first integer which remains is 3; now in the remaining sequence we shall strike off every term whose index is a multiple of 3. In the sequence which remains now and which consists of 1, 3, 7, 9, 13, 15, 19, ..., the first term which has not been used already is 7. We shall, therefore, strike off every term in this sequence whose index is a multiple of 7; that is to say, the 7th (which is 19), 14th, 21st, etc., term in this sequence. In the remaining sequence we shall look up the first term which has not been used (it is 9) and again strike off terms whose indices are multiples of it. We continue this process indefinitely; it is obvious again that infinitely many terms remain. They may be called the result of our sieve.

The aim of our exercise was to consider certain asymptotic properties of this latter sequence of numbers, let us call them for brevity "lucky numbers." All lucky numbers up to 48,000 were quickly computed on the machine and the following data about them were obtained. (For the first few see Table I.)

1) The number of lucky numbers between 1 and n seems to compare quite well with the number of primes from 1 to n. We append a short table of their frequencies (Table II).

2) We noticed the number of luckies of the form $4n + 1$ and of the form $4n + 3$. This ratio seems to tend to 1 with a preponderance, at first, of the luckies of the form $4n + 3$. More generally we looked at the number of luckies of the form, say, $5n + \alpha$, $\alpha = 0, 1, \ldots 4$, etc. It is obvious that there are no luckies of the form $3n + 2$.

4) More generally we looked at gaps of a given length between successive luckies. It seems that as far as we went the number of gaps $N(k)$ of a given length k corresponds to the number of gaps $P(k)$ of the same length between successive primes.

5) Every even integer between 1 and 100,000 is a sum of two luckies.

6) There are 715 numbers which are simultaneously prime and lucky between 1 and 48,600.

More detailed tables exist on magnetic tapes.

The similarity between the behavior of the lucky numbers and that of primes seems to be rather striking in the range of the integers which we have considered. Obviously it is rather difficult to prove general theorems. For example, the question of whether there are infinitely many primes among the luckies is certainly difficult to answer. The sieve defining the lucky numbers can be written down as follows: let us define a sequence of sequence of integers

$$a_n^{(1)} = n;\ a_n^{(2)} = \{a_{n\,v}^{(1)}\} - \{a_{2v}^{1}\};\ a_n^3 = \{a_n^{(2)}\} - \{a_{v\cdot a_2^2}^2\}$$

$$a_n^l = \{a_n^{l-1}\} - \{a_{v\cdot a_l^{l}}^{l-1}\},\ v = 1, 2, 3, \ldots$$

(All our sequences on the left hand side consist of the sets as given by the right hand side in natural order.) The lucky numbers are simply $a_k^{(k)}$. One could consider other variations of the sieve. It is proposed to make a few more experiments on numbers resulting from such sieves.*

* Some years ago one of us (S.U.) has discussed sieves of this kind with P. Erdős, Professor D. H. Lehmer has mentioned that Erdős recently obtained some general results on a sieve of this sort.

TABLE I

FIRST 45 LUCKY NUMBERS

1	3	7	9	13
15	21	25	31	33
37	43	49	51	63
67	69	73	75	79
87	93	99	105	111
115	127	129	133	135
141	151	159	163	169
171	189	193	195	201
205	211	219	223	231

Between 1 and 48,600 there are 715 numbers which are simultaneously prime and lucky.

TABLE II

	LUCKIES IN INTERVAL N		PRIMES IN INTERVAL N	
N	No. in Interval	Total thru N	No. in Interval	Total thru N
1-2,000	276	276	304	304
2,000-4,000	227	503	247	551
4,000-6,000	213	716	233	784
6,000-8,000	204	920	224	1008
8,000-10,000	198	1118	222	1230
10,000-12.000	195	1313	209	1439
12,000-14,000	188	1501	214	1653
14,000-16,000	196	1697	210	1863
16,000-18,000	183	1880	202	2065
18,000-20,000	186	2066	198	2263
20,000-22,000	186	2252	202	2465
22,000-24,000	181	2433	204	2669
24,000-26,000	179	2612	192	2861
26,000-28,000	178	2790	195	3056
28,000-30,000	180	2970	190	3246
30,000-32,000	173	3143	187	3433
32,000-34,000	177	3320	206	3639
34,000-36,000	168	3488	186	3825
36,000-38,000	179	3667	193	4018
38,000-40,000	173	3840	186	4204
40,000-42,000	170	4010	189	4393
42,000-44,000	178	4188	187	4580
44,000-46,000	160	4348	182	4762
46,000-48,000	175	4523	185	4947
48,000-48,600	48	4571	53	5000

TABLE III

NUMBER OF GAPS OF LENGTH *K*

K	*N*(*k*)	*P*(*k*)
2	647	680
4	621	677
6	824	1075
8	351	411
10	361	478
12	509	517
14	184	238
16	172	168
18	267	253
20	106	105
22	112	101
24	130	77
26	32	34
28	51	38
30	66	65
32	21	12
34	24	15
36	33	20
38	13	4
40	9	7
42	10	5
44	4	3
46	5	0
48	1	1
50	2	2
52	6	4
54	2	3
56	4	0
58	1	1
60	0	1
62	0	1
64	1	0
66	0	0
68	0	0
70	0	0
72	0	1

TABLE IV

n	LUCKIES OF THE FORM $4n+1$, $4n+3$		PRIMES OF THE FORM $4n+1$, $4n+3$	
	A_1^4	A_3^4	A_1^4	A_3^4
1-2,000	133	143	148	155
2,000-4,000	102	125	122	125
4,000-6,000	108	105	114	119
6,000-8,000	97	107	116	108
8,000-10,000	99	99	110	112
10,000-12,000	93	102	98	111
12,000-14,000	96	92	111	103
14,000-16,000	102	94	102	108
16,000-18,000	96	87	97	105
18,000-20,000	88	98	108	90
20,000-22,000	95	91	98	104
22,000-24,000	96	85	104	100
14,000-16,000	99	80	96	96
26,000-28,000	95	83	99	96
28,000-30,000	98	82	89	101
30,000-32,000	92	81	94	93
32,000-34,000	109	68	105	101
34,000-36,000	95	73	86	100
36,000-38,000	89	90	99	94
38,000-40,000	90	83	90	96
40,000-42,000	90	80	92	97
42,000-44,000	82	96	97	90
44,000-46,000	75	85	93	89
46,000-48,000	81	94	92	93
48,000-48,600	19	29	27	26
Total	2319	2252	2487	2512

TABLE V

n	LUCKIES OF THE FORM $5n+1$, $5n+2$, $5n+3$, $5n+4$, $5n+5$					PRIMES OF THE FORM $5n+1$, $5n+2$, $5n+3$, $5n+4$, $5n+5$				
	A_1^5	A_2^5	A_3^5	A_4^5	A_5^5	A_1^5	$AA_2^5\ A$	A_3^5A	A_4^5	A_5^5
1-2,000	60	50	56	60	50	74	78	78	73	1
2,000-4,000	47	44	52	39	45	61	64	60	62	
4,000-6,000	42	41	49	50	31	60	57	61	55	
6,000-8,000	51	37	40	41	35	52	58	57	57	
8,000-10,000	44	42	42	34	36	60	52	54	56	
10,000-12,000	45	39	40	38	33	47	54	53	55	
12,000-14,000	36	45	39	41	27	52	55	54	53	
14,000-16,000	38	43	40	38	37	55	55	1	49	
16,000-18,000	35	44	36	28	40	51	52	49	50	
18,000-20,000	35	37	38	34	42	52	45	52	49	
20,000-22,000	39	35	37	36	39	50	52	50	50	
22,000-24,000	43	36	30	39	33	52	51	49	52	
24,000-26,000	37	38	52	33	29	45	45	52	50	
26,000-28,000	36	35	36	30	41	52	54	47	42	
28,000-30,000	36	33	35	36	40	46	47	48	49	
39,000-32,000	38	34	36	35	30	46	49	46	46	
32,000-34,000	30	37	39	31	40	49	47	56	54	
34,000-36,000	40	38	41	24	25	50	44	46	46	
36,000-38,000	30	34	34	40	41	44	55	49	45	
38,000-40,000	43	31	32	30	37	47	38	49	52	
40,000-42,000	39	30	36	31	34	43	45	51	50	
42,000-44,000	35	26	40	42	35	47	47	51	42	
44,000-46,000	43	20	31	35	31	44	47	46	45	
46,000-48,000	34	35	29	37	40	51	49	36	49	
48,000-48,600	9	8	6	13	12	13	12	14	14	
Total	965	892	936	895	883	1243	1252	1259	1245	1

Los Alamos Scientific Laboratory

COMMENT ON THE F.L.T. ROUND TABLE: II

H. W. Becker

Heimann & Elston [10] certainly injected a new idea into the problem, I found more stimulating and valuable than if it were merely 100% accurate. (Reviewing the note backward) they conclude "the number of Pyth. triples in the series from 1 to $2n^2$, $W_2 = (n-1)/\sqrt{8}$."

Let $n = Z/\sqrt{2}$, and let $W(Z)$ be the number of P $\triangle$ of hyp. $\not>$ Z. Then dropping the -1 as negligible, the authors' conclusion is that

(2) $$W_{H-E}(Z) \approx Z/4\,,$$

which is historically important as the first attempt at approximating $W(Z)$. To give it a fair test, I extended my table of P $\triangle$ from $Z = 225$ to $Z = 450$. I find $W(450) = 337$, or just 3 times their estimate of $450/4 = 112$.

Let $W^*(Z)$ and $W'(Z)$ be respectively the number of primitive and non-prim. P $\triangle$ of hyp. $\not>$ Z. Then by Lehmer [11], very closely,

(3) $$W^*(Z) \approx Z/2\pi\,.$$

I owe Heimann & Elston the stimulus to the exact formula

(4) $$W(Z) = W^*(Z) \div W^*([Z/2]) \div \ldots \div W^*([Z/\{Z/5\}]) = \sum_{i=1}^{Z/5} W^*([Z/i]),$$

where [ ] denotes the greatest integer function. Thus $W(50) = W^*(50) \div W^*(25) \div W^*(16) \div W^*(12) \div W^*(10) \div W^*(8) \div W^*(7) \div W^*(6) \div 2W^*(5) = 7 \div 4 \div 2 \div 7\cdot 1 = 20$. It is well known that

(5) $$-(\log z) + \sum_{i=1}^{z} 1/z \approx \gamma = 0.5772157\ldots = \text{Euler's const.},$$

therefore

(6) $$W(Z) \approx W^*(Z) \sum_{i=1}^{Z/5} 1/i \approx Z\,\{\delta \div \log\,(Z/5)\}/2\pi$$

where δ is of the order of γ, but perhaps oscillates about or converges to $\gamma/3$. Thus $\delta = 2\pi/5$ at $Z = 5$, but 0.2 at $Z = 50$ and 450. Consequently, neglecting δ, *the authors'* (2) *above is too small by a factor of about* $(2/\pi)\log(Z/5)$.

Whence this discrepancy? To begin with, the authors allow the hyp.

to be $\leq n\sqrt{2} = Z$, but require the legs to be $\leq n = Z/\sqrt{2}$. This excludes all P △ with a leg $> Z/\sqrt{2}$, whereas it is well known that there is an infinity of P △ with a leg differing from the hyp. only by unity, or by other small constants. This type of exclusion is familiar to electronics instructors, who start off the theory of tuning and filtering by saying that the edge of the band-width is where the response is down $1/\sqrt{2}$ (i.e. 3 decimals) from peak.

Which brings to light another surprise. Denote by $W''(Z)$ the number of P △ of hyp. $\not> Z$, whose longest leg $\not> Z/\sqrt{2}$. Then almost exactly, for all Z tabulated,

$$W''(Z) = W'(Z), \tag{7}$$

e.g. $W''(450) = 261$, $W'(450) = 265$ instead of 112. So at $Z = 450$, this exclusion only accounts for about 1/3 of the discrepancy, from 337.

Going back to first principles, due to Fermat [4] p.227, we see that those integers $\not> 2n^2 = Z^2$ which have no prime factor of form $4k \div 1$ have no hope of being hyps. of P △ , so the authors' denominator is much too large. On the other hand, where there is such a factor, the integer is certain to be such a hyp.; and the weighting function for the number of choices of legs will not be a binomial coefficient, but will depend on the number of such prime factors of the integer, so the authors' numerator is much too large also. Still, since without their bold unverified conjecture the more precise (6) would not be known at this time, they are entitled to a full share of credit in the math. teamwork (an assist, so to speak, for starting the triple play yet to be finished!) And if we assume that their formulas actually enumerate only the primitive instead of all solutions, then W_2 is very good, too large by a factor of only $\pi/2$.

COMMENTS ON THE COMMENT

Fred G. Elston

Dear Dr. James:

Thank you very much for your sending me the paper of Mr. Becker. I have read it with the greatest of interest and would like just to make a few comments.

The remarks about finding Pythgorean triples seem to me correct but they do not touch our problem on hand.

We did not claim that the formula for W gives the exact number of cases in the given area when n becomes very high. We simplified the formula right in the beginning to our disadvantage, because all we wanted to show was that with increasing r the value of W decreases and with the increasing n the same thing is happening. This latter fact gave me the idea of a contradiction between the density of "cases" for r larger than 3 from the probability point of view and the quite different rational density from the analytic point of view. Whether this contradiction leads to the conclusion: "there are no such number triples" or "there are in all probability no such number triples" that question is still wide open for discussion.

It seems that Mr. Becker on page 4, first paragraph has misunderstood our argument. The values of x and y are limited by 1 and n. The possibility of a "hit" z reaches not quite up to $\sqrt{2}n$. We did not claim that there could not be values for x, y, and z between n and $\sqrt{2}n$ but this does not take part in our argument because we took just the numbers from 1 to n under consideration as far as the choice of elements in our combination is concerned. We have to distinguish between "elements" x and y and "hit points" z. Here is maybe the difference between our "Formula" and the practical results for Pythagorean numbers.

Fred G. Elston

Errata in "*Proof of F.L.T. For All Even Powers*" **by H. W. Becker**

(see Mathematics Magazine, May.-June, 1955)

Page 297, 5th paragraph, last line, should read:

$$(U+V)_e^{2m},\ Y = (U+V)_o^{2m}.\ \text{Or } Z,Y = [(U+V)^{2m} \pm (U-V)^{2m}]/2$$

The next paragraph, add: implicitly.

The last paragraph on p.297, add:

Somewhat similiarly it is shown that if z^m, y^m (or x^m, y^m) are $P\triangle$ sides then z, y (or x, y) can't be $P\triangle$ sides.

Dear Mr. Craig,

H. W. Becker in his comments on L. Caner's "Pythagorean Principle and Calculus", MATHEMATICS MAGAZINE (28 AND 29, 1955) states that "the author has assumed uz^{-2} a constant in advance of proving it". I should like to see a more complete discussion on this point, for to me it seems that (uz^{-2}) is shown to be a constant from the fact that the equal quantity, $(\sin^2\theta + \cos^2\theta)$, has a derivative equal to zero.

Very truly yours,

R. T. Coffman

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

College Algebra and Trigonometry. By Frederic H. Miller, John Wiley & Sons, 440 Fourth Ave. New York 16, 342 pages, $4.50.

This new book is now available in a second edition. Published October the new book provides a unified treatment of the basic principles needed by college students.

Dr. Miller, who has retained the general content, methodology, and organization of his book, has introduced a number of changes into the second edition. Foremost among these is a nearly complete turnover in the exercises which now number over 2,100. Besides these replacements, he has inserted a list of supplementary exercises at the end of each chapter, designed to stimulate new concepts and methods, and to give further choice and a more extensive review.

Richard Cook

Prelude To Mathematics. By W. W. Sawyer, Penguin Books Inc., Baltimore, Md., 1955, 65¢.

In *Mathematician's Delight* one of the most popular Pelicans so far published, W. W. Sawyer described the traditional mathematics of the engineer and the scientist. In this new book the emphasis is not on those branches of mathematics which have great practical utility, but on those which are exciting in themselves: mathematics which is strange, novel, apparently impossible, for instance an arithmetic in which no number is bigger than four. These topics are preceded by an analysis of that enviable attribute "the mathematical mind." Professor Sawyer not only shows what mathematicians get out of mathematics, but also what they are trying to do, how mathematics has grown, and why there are new mathematical discoveries still to be made. His aim is to give an all-round picture of his subject, and he therefore begins by describing the relationship between pleasure-giving mathematics and that which is the servant of technical and social advance.

Elizabeth Creak

The Bequest of the Greeks. By Tobian Dantzig, Charles Scribner's & Son's, New York, 1955, $3.95.

"Professor Dantzig has justly acquired the reputation of being one of the foremost interpreters to the layman of the beautiful and profound, but relatively unvisited, world of mathematics. His famous book *Number: The Language of Science* (published some years ago) has been translated into almost all the major languages of the world, and is now in its fourth edition after numerous reprintings.

"In this book the author returns to his favorite theme by tracing the evolution of mathematics from the 6th century B.C. through the last major contributions of the ancient Greeks. It might very properly be called the "human story of mathematics," for mathematics is a science whose development took enormous forward steps as the result of lightning flashes of insight in the minds of certain exceptionally gifted individuals. How did the Greeks come to organize into an elegantly abstract deductive system the various geometric rules-of-thumb of their precursors? How did some of the famous mathematical brainteasers of antiquity lead to the development of whole new branches of mathematics? These are some of the questions upon which Professor Dantzig has based this lively history."

Editor's note: Chapter six is an unfortunate low in this excellent book. We have published good papers (one of our best on Fermats Last Theorem) by "pseudomath's." I encourage anybody and everybody in their work on anything in mathematics.

Shop Mathematics. By Claude E. Stout, John Wiley & Sons., New York, 1955, 282 pages, $3.70

Published in October, this new book concentrates on principles, particularly as they apply to situations that confront the craftsman on his job.

Covering the fundamentals of arithmetic, algebra, geometry, trigonometry, logarithms, and the slide rule, Professor Stout also includes many shop applications. His emphasis is on accuracy of computation, and it is toward this end that he gives the methods for checking most operations.

Richard Cook

Plane Algebraic Curves. By E. J. F. Primrose, St. Marten's Inc. New York, $3.00.

Plane Algebraic Curves is based on lectures which I have given to honours mathematics students at the University College of Leicester during the last few years. Its aim is to give the student who is unfamiliar with the theory of curves a reasonably brief introduction to the subject: he should then be in a position to study the more advanced works, some of which are recommended at the end of the book.

It may fairly be said that there is a need for such a book. There is no elementary book on curves which is still in print, and works such as Coolidge's *Algebraic Plane Curves* and the recent book by Walker on *Algebraic Curves*, are too difficult for a beginner.

E. J. F. Primrose

Science Awakening. By Prof. Dr. B.L.v.d. Waerden of the Swiss Federal Institute of Technology. Translated from the original Dutch text by Prof A. Dresden. 312 pages with 28 half-tone plates and numerous test-illustrations. P. Noordhoff Ltd., $5.50.

The history of the exact sciences is part of the history of culture, it is in fact one of its most important components, for nowadays our entire civilization is governed by science. The most beautiful thing the Greeks have bestowed upon us is undoubtedly their art, but their mathematics has turned out to be the most vital, since our very life and death depend on the mathematical sciences.

This aspect of Greek thought is not brought sufficiently to the fore in books on the history of culture.

This volume is an attempt to make good this deficiency. It is not only meant for mathematicians, but for all educated people, educated in the sense that they can understand school mathematics and are interested in it; more is not required.

If a mental picture of Greek mathematics is to be obtained, the entire background of Greek culture, philosophy, astronomy, and life in Greek cities and at the royal courts of Alexandria and Syracuse should first be seen. In fact, one should travel still further back in time.

An altogether new light is thrown upon Greek mathematics by ancient Babylonian cuneiform texts in which geometrical and algebraic problems are solved with astonishing dexterity.

The publishers

Basic Mathematics for Science and Engineering. By Paul G. Andres, Hugh J. Miser, and Haim Reingold, John Wiley & Sons, Inc., New York, 1955,

A practical text, the new book is also designed as a refresher for the practicing engineer, technician, and scientist. The book assembles those topics from algebra, trigonometry, and analytic geometry that are essential to the study of calculus, engineering theory and practice, and the various sciences. Among the outstanding features are early explanations of the slide rule and trigonometic functions, the emphasis on graphical methods, and the treatment of the Doolittle method of solving simultaneous linear equations. Over 650 examples are worked out in the course of the volume, and more than 7,000 exercises serve to emphasize principles of accuracy and numerical computations.

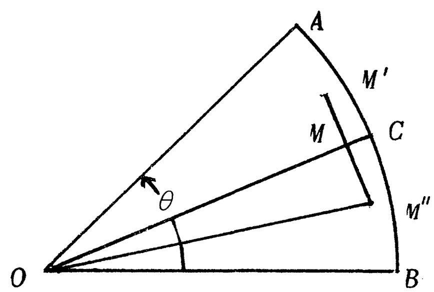
Richard Cook

AN EXAMPLE IN FUNCTIONAL EQUATIONS

R. Steinberg

The following example may be of interest to some teachers. It indicates that, in finding centers of mass, it is not always necessary to use integration - rather, the limiting processes involved are those of differentiation. We take as proved the fact that, if a plane region or arc is contained between two parallel lines, then so is the center of mass.

Now, let us consider the problem of finding the center of mass M of an arc AB of the unit circle which subtends an angle θ at the center O of the circle. By symmetry we know on which radius vector M lies and we wish to find the distance $f(\theta)$ of M from O. If we



divide the arc AB into two equal parts at C and call M', M'' the centers of mass of the arcs AC, CB respectively, then M is the mid-point of $M'M''$, by symmetry. But, $OM = f(\theta)$, $OM'' = f(\theta/2)$ and $\angle M'OC = \theta/4$. Thus we get our functional equation:

$$f(\theta) = f(\theta/2) \cos \theta/4. \qquad (1)$$

In order to solve (1), we first consider the situation at $\theta = 0$. Since the arc AB is contained between the lines AB and the tangent line at C, it follows that M is also, and hence that

$$\cos \theta/2 \leq f(\theta) \leq 1.$$

Now, if we let $\theta \to 0$, we get $\lim_{\theta = 0} f(\theta) = 1$.

Now to solve (1) we rewrite it and then iterate:

$$f(\theta) \cdot (\theta/2)/(\sin \theta/2) = f(\theta/2) \cdot (\theta/4)/(\sin \theta/4)$$

$$= f(\theta/2^n) \cdot (\theta/2^{n+1})/(\sin \theta/2^{n+1}),$$

by induction.

Now, let $n \to \infty$. Then, the right side approaches 1.

Thus,

$$f(\theta) = (\sin \theta/2)/(\theta/2).$$

In closing, we remark that, as long as there is circular symmetry, the same functional equation and method of solution apply; for example, an arc may be replaced by a sector or by the portion of a surface of revolution contained between meridians. The only distinction of the various cases is the determination of $\lim_{\theta \to 0} f(\theta)$.

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POPULAR PAGES

A GRASS-ROOT ORIGIN OF A CERTAIN MATHEMATICAL CONCEPT

Glenn James

When formulating abstract concepts we focus our attention on some particular property common to several individual (concrete) cases and ignore the distinctive properties of these cases. For instance if we are defining the abstract concept denoted by 2, we may note that a couple of roses, a couple of sparrows, a couple of mountains, or what not, all have this twoness property, and we ignore those properties which make "a rose a rose," a sparrow a sparrow, or a mountain a mountain, for these properties do not enter into the concept, 2. In broader terms the universe is an aggregation of widely different elements and mathematics is the story of the harmonies between them; or perhaps we should say the study of certain principles that are common to them. Unfortunately even the history of mathematics tells us too little about how these principles have been and are being discovered, andcan be rediscovered by the uninitiated. Certainly some of them have arisen from attempts to explain puzzlements that have arisen in our workaday lives. Such is the case in the episode that we shall now relate.

Long years ago, when selling peaches on my father's farm, we found that people would pay nearly as much for a bushel of small peaches as for the same measure of exceptionally large ones. But one day a very wealthy housewife with a reputation for being a "tight wad" paid three times as much for very large peaches as she would have had to pay a competitor for the same amount of very small ones. Seeking an explanation of this paradox, we measured several widely different sized peaches, then peeled and seeded them and measured the residues. We thus found that if the peaches were treated as perfect spheres, the edible portions varied directly as the diameters, with only a small tolerance. This seems reasonable because in a bushel of peaches with diameters half that of larger ones there are eight times as many pits and twice as much peel. Obviously this result is valid for any fruit that has pits or cores and is more or less spherical; plums, apricots, apples, pears, etc.

At this point we mathematicise the problem by assuming that we are dealing with abstract spheres, and simplify it by neglecting the pits. Suppose we start with one spherical ball 16″ in diameter in a cubical box with edges 16″. Then replace it with balls 8″ in diameter.

Obviously eight of the small balls would be required to fill the box. Since the volume of a sphere varies directly as the cube of its diameter (i.e. $v = 1/6\pi D^3$) the volume of each small ball is 1/8 that of the large one. So of course the total volume of the eight small balls is exactly equal to the volume of the large one. But the surface of a sphere varies directly as the square of its radius (i.e. $S = \pi D^2$). Hence the surface of each small ball is one fourth of the surface of the large one and the combined surfaces of the eight small ones is twice the surface of the large one. Thus by filling the box with balls half as large in diameter we do not change the total volume of the balls in the box but double the surface. Continuing this halving process the total volume always remains the same, while the surface doubles each time so increases "without limit" (is infinite*).

Suppose now that each time we halve the diameter we take out one half of the balls. Then the total surface remains the same but the total volume is decreased one half, in each step. Hence the successive total volumes are 1/2, 1/4, 1/8, ... of the original volume, and this sequence approaches zero. In elite mathematics, we would say that the limit of the piles of balls is the bottom of the box but the limit of the sum of the surfaces of these sets of balls is not the surface of the bottom of the box, it is indeed a constant dqual to the surface of the one large ball.

Again suppose we halve the diameters, successively, as above, but remove only one fourth of them at each step. Then the total volumes run 3/4, $(3/4)^2$, $(3/4)^3$, ... of the original volume, and the total surfaces run 3/2, (3/2) , (3/2) ... of the original surface. Thus the pile of balls approaches the bottom of the box but the total surfaces of the sets of balls is infinite (increases without limit). We leave here the problem of what happens to the total surfaces if other fractional parts of the total volumes are removed.

Now it happens to be true in general that *similar volumes vary as the cubes of corresponding dimensions*, and surfaces of such solids *vary as the squares of corresponding dimensions*. So if a farmer has a hill on his farm, he can replace it by four similar ones with dimensions one half those of the large one, haul off the other half of the dirt, and have the same amount of surface that he had before. Repeating this process he would, after *sufficient* time, obtain what would appear to be a level field with just as much area as he had with the original hill.

There is quite a bit to think about here if he removes less than half of each set of hills.

Obviously some such situations as the above could arise without the successive sets of solids being similar. Similarity might be

* In the way of application, this shows why we grind hard materials into a powder to make them dissolve rapidly.

replaced by other restrictions which would make the total volumes of the sets approach zero but the total surface remain constant or mayhap increase or decrease as one desires.

But the simplified problem stirred up by the pigmy peach and the giant peach has gone far enough to illustrate how advanced mathematical concepts can grow out of grass-root origins.

* * * * * *

Just after the above article was finished I received the following note through a friend of its writer. "The writer is a PhD in science from a leading university."

> "I have a pet peeve (one of many) about the initial contact the child has with numbers. We teach a youngster to count marbles, pencils, or safety pins - real things. Then when he starts to school he is abruptly introduced to abstractions in his combinations. Three plus two. Not three pencils and two more. Just three plus two. Do you suppose the difficulty many students, say medical students, have in solving real problems is conditioned by this early hiatus between the abstract and the concrete? I maintain that you can't add abstractly at all; that 3 and 2 make 5 only if the 3 and the 2 are something, not nothing. (Three nothings are zero!) My 16 year old daughter pointed out to me that only mouns can be added, not adjectives. Since three and two started out as adjectives it makes just as much sense, fundamentally, to add yellow and ugly, as to add three and two abstractly."

I have no doubt that a great deal of students' troubles in mathematics are, "Conditioned by this early (mistaken, Ed.) hiatus between the abstract and concrete." The abstract numbers 2 and 3 are not "nothings." They are indeed *something*. They represent the numbers in *their kinds of sets* regardless of what the individuals are in these sets.

In fact probably the best way to teach the meaning of numbers is to start children counting with sticks, then substitute various things for the sticks, one at a time, until he sees that counting is entirely independent of the nature of the individuals, is *dependent only on the "manyness" of them.*

When you add the two numbers, 2 and 3, you have the numbers in two sets combined making 5, regardless of the natures of the individuals. two cows and three horses make five, a *set of five*. Whether you call them cows and horses or just animals is a matter of biology and has nothing whatever to do with the addition. *The addition holds no matter what individuals make up the groups. Therein lies the power and granduer of mathematics*. When we do attach special meanings to numbers, we call the results, "APPLIED MATHEMATICS."

SUGGESTIONS TO PROSPECTIVE AUTHORS

A subscriber writes: "I notice that MATHEMATICS MAGAZINE is soliciting contributions for the Semi-popular and Popular Pages. Since many contributors may be unfamiliar, as I am, as to the time required to process a contribution to the point of acceptance or rejection it would be helpful to know what is to be expected in this matter. I suggest that a brief statement covering this be published in the magazine."

The time required "for processing" depends upon the paper and if it is technical upon the referee or referee's, who are always busy with their own work and do refereeing for the love of the cause. Some referees may return a paper in a few weeks others in a year or more. A paper that is poorly gotten up, is not clear, usually requires more time for refereeing especially if a first referee is undecidedor unfavorable and we feel that another referee should read the paper. If the two differ it may be necessary to send it to a third. If it is then accepted except for re-writing, it must go back to the author who may keep it a long time.

Author's can speed up the processing of their papers by:

(1) introducing them with a carefully written foreword stating their purposes and contents in brief, non-technical language and indicating what is new if it be a research paper.

(2) making them self-contained, i.e. making what purports to be proofs complete proofs using references only as supports or as settings but *not as steps in the proofs.* (Generally old material should be presented descriptively, sources being cited if it is not pretty well known).

(3) writting papers for your audience. What this means can be understood by reading some papers already published in the department for which you intend yours. For example, if you are intending your paper for the Semi-popular and Popular Pages it would be well to read, or re-read, Vol. 28, No. 1, pp. 39-43; No. 3, pp. 173-176; No. 4 pp. 208-220; No. 5, pp. 299-303 and Vol. 29, No. 1, pp. 89-99; this current issue, pp. 131-133.

(4) having some one read and critise their papers before submitting them for publication. This is probably not so necessary for well-experienced writers but they are just the ones who are most likely to take this most important precaution.

Editor

A DEVELOPMENT OF ASSOCIATIVE ALGEBRA AND AN ALGEBRAIC THEORY OF NUMBERS, III

H.S. Vandiver and M.W. Weaver

Introduction

The first paper under the above title (which will be called (I)) appeared in this magazine, Vol. 25, 233-250 (1952), and the second paper under the same title appeared also in this magazine, Vol. 27, 1-18 (1953) (which will be called (II)), each under the authorship of Vandiver. In (I) the foundations of ordinary algebra were developed from a standpoint which also yielded the foundations of an infinity of finite algebras.

In (II) polynomials in x, an indeterminate, were defined involving natural numbers only. Congruences involving these polynomials were then shown to justify the adjuction of zero, the negative integers and the rational fractions to the set of natural numbers. Finally it was shown how the methods could be extended easily so as to provide a foundation for the classical theory of algebraic numbers.

In the present paper the writers treat the theory of semi-groups. To lead up to the abstract theory, an introductory account of which is given, semi-groups formed by substitutions or correspondences are considered. In the usual theory of substitutions an expression like

$$\text{(A)} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 6 & 3 \end{bmatrix}$$

is used to mean an operator which substitutes 2 for 1, 1 for 2, 4 for 3, 5 for 4, 6 for 5, and 3 for 6. But suppose we allow repetitions in the second line, such as in

$$\text{(B)} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 4 & 6 & 6 \end{bmatrix}$$

then we are led into a theory which is far more general than the theory based on the substitutions of the type (A) where the second line contains no repetitions. If we define multiplication of substitutions in the usual way, then Cayley's theorem tells us that any finite group can be represented by a set of substitutions of the (A) type. *A wide generalization of Cayley's theorem is due to R. R. Stoll*[1], *who has proven that any finite semi-group can be represented by a set of gen-*

[1] "Representations of Finite Simple Semi-groups", *Duke Math Journal* 11, 251-265, (1941).

eralized substitutions of the (B) *type.* (Generalized substitutions seem to have been first mentioned, in the literature, by Suschkewitsch (*Math Annalen* 99, 30-50 (1928)), but he did not treat them with much detail.) *In our present paper we begin a detailed examination of the structure of generalized substitutions, we think, for the first time.*

Throughout this article we may state well known theorems without proofs, particularly when proofs may be found in a number of books on algebra. Some of these are stated as problems. As far as demonstrations are concerned we devote nearly all our attention to new results. New concepts, however, will be discussed at length, with many examples given.

The work on this paper was done under National Science Foundation Grant G1397.

We are indebted for suggestions on and corrections to the material in this paper to F. C. Biesele, Ann Breese Barnes, J. D. Buckholtz, J. L. Dorroh, O. B. Faircloth, R. P. Kelisky, H. C. Miller, A. M. Mood, C. A. Nicol, T. P. Pitts, D. K. Riddle, J. M. Slye, P. H. Thrower, and W. J. Viavant. J. D. Buckholtz assisted particularly in setting up the definition of excycle and provided a neat proof of problem 7.

* * * * * * * *

ASSOCIATIVE ALGEBRAIC SYSTEMS OF SINGLE COMPOSITION

Sets. We do not attempt to define the words *set* and *element*; however, we speak of a set S of elements only when it is true that the elements of S are so classified that no two elements of S fall into the same classification. We do not wish to assume that a set must contain at least one element. In fact we may speak of the null set as being a set which contains no element and we denote it by O. We use the following abbreviations and definitions:

$n \in N$ means that n is an element of the set N.

$N \subseteq M$ means that each element of the set N is an element of the set M. In this case we say that N is a *subset of* M. The same meaning is assigned to $M \supseteq N$.

$N = M$ means that both $N \subseteq M$ and $N \supseteq M$.

$N \subset M$ means that $N \subseteq M$, but some element of M is not an element of N. If such is the case, we may also write $M \supset N$. Whenever $N \subset M$ we call N a *proper subset of* M.

$P = M \cap N$ means that $p \in P$ if and only if both $p \in M$ and $p \in N$.

$Q = M \cup N$ means that $q \in Q$ if and only if either $q \in M$ or $q \in N$.

If a is not an element of S, we write $a \notin S$; and similarly if one of the statements $M \supseteq N$, $M \subseteq N$, $M \supset N$, $M \subset N$ is not true, we draw a line through the middle symbol of that statement.

Suppose that we start with a non-null set T and that we divide T into a set T^* of non-null subsets $A, B, \ldots,$ of T having the property: if $m \in T$, then m is contained in one and only one of the above subsets. Then we have *partitioned* T into a set T^* of subsets of T, and these subsets are a *partitioning* of T into subsets. We shall use the equivalence relation of an algebraic system as such a partitioning of a given set; so that we can think of such a system as being either a set T of elements such that several elements may be equivalent, or as being a set T^* of subsets of T such that no two subsets are equivalent.

Some generalizations of the definitions of the theory of substitutions.

Let two non-null sets R and S be given. The statement that M is a *mapping of R into S* means that M is an operator such that if $a \in R$ then M relates a to exactly one $a' \in S$. This may be described by $M[a] = a'$. We shall consider *operator* and *relates* as undefined. Let M also satisfy: if $b' \in S$, then there exists a $b \in R$ such that $M[b] = b'$; then M is said to *map R onto S*. We may describe this by $M[R] = S$. A mapping of a finite set N into itself is called a *correspondence on* N. A correspondence P on N such that P maps N onto itself satisfies the idea of substitution given in (I) and will be given that name in this paper. As in substitution group theory, (Cf. (I), pp. 239-241), if C is a correspondence on N: $a_1, a_2, \ldots, a_n$, and $C[a_i] = b_i$, for $i = 1, 2, \ldots, n$, C may be denoted by

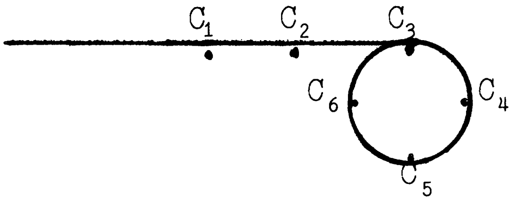
$$\begin{bmatrix} a_1 & a_2 \cdots & a_n \\ b_1 & b_2 \ldots & b_n \end{bmatrix};$$

here the parentheses may be thought of as the operator and a_i is related to b_i by being written above it. (We assume that the reader knows a little about the theory of substitution groups.) If $M[a_i] = a_i$ sometimes we omit a_i from the parenthesized expression above. We may have repeated elements in the line of b's. Examples below will be numbered E1, E2, etc., while problems will be numbered 1, 2, etc.

E1. Mapping of R into S. Let R be the set of natural numbers and S be the set containing only the indeterminate x used in adjoining O in (II). Let righthand multiplication by x, designated by x written on the right of a natural number, be the operator and congruence modulo x be the relation; then the operator relates each element of R to some element of S and therefore maps R into S.

E2. Mapping of R onto S. The above example is also an example of an "onto" mapping since for each $b' \in S$ there exists a $b \in R$ such that right-hand multiplication by x relates b to b'

E3. Correspondence on N. Consider the diagram



of (I), and let an operator M move C_i into the position of C_{i+1} for $i = 1, 2, 3, 4, 5$, and move C_6 into the positon of C_3, and let $M[C_i] = C_{i+1}$ mean that C_{i+1} is replaced by C_i in the diagram for $i = 1, 2, 3, 4, 5$ and $M[C_6] = C_3$ mean that C_3 is replaced by C_6; then M is a correspondence on the set of C's, and may be described by

$$\begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ C_2 & C_3 & C_4 & C_5 & C_6 & C_3 \end{bmatrix}.$$

We continue with our definitions. If M is a correspondence on a set N, $a \in N$, and there exists an element b, not a, of N such that either $M[a] = b$ or $M[b] = a$, then a is *involved in* M; otherwise a is *not involved in* M. If $N' \subseteqq N$ and N' contains exactly r elements, then N' is said to be of *degree* r. If C_1 and C_2 are two correspondences on N then C_1 is said to be *equivalent to* C_2, written $C_1 \cong C_2$, if for each $a \in N$, $C_1[a] = C_2[a]$ Otherwise we say that C_1 is *not equivalent* to C_2 and write $C_1 \ncong C_2$. It is clear that either $C_1 \cong C_2$ or $C_1 \ncong C_2$. The *product* $C_1C_2 \ldots C_r$ of a finite set of correspondences is defined as being equivalent to the correspondence C_m such that

$$C_m[x] = C_r\left[C_{r-1}\left[\ldots\left[C_1[x]\right] \ldots\right]\right],$$

where $r > 1$, and x varies over N. We postulate that products equivalent to equivalent correspondences are equivalent to each other. The reader may verify that each product of a finite set of correspondences exists. A parenthesized product means the correspondence which the product inside the parenthesis is equivalent to. We notice that the *associative law* holds, that is, $C_1(C_2C_3) \cong (C_1C_2)C_3$, since

$$\begin{aligned} C_1(C_2C_3)[x] &= (C_2C_3)\left[C_1[x]\right] \\ &= C_3\left[C_2\left[C_1[x]\right]\right] \\ &= C_3\left[(C_1C_2)[x]\right] \end{aligned}$$

$$= (C_1C_2)C_3[x]$$

for each $x \in N$. Only natural numbers are used for exponents, and C^n means a product of n factors, each of which is equivalent to C. If S is a set of correspondences on N, such that $C_1C_2 \in S$ and $C_2C_1 \in S$ whenever $C_1 \in S$ and $C_2 \in S$, then S is called a *semi-group of correspondences on N.*

E4. Elements involved in a correspondence. The elements involved in the correspondence

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 3 & 3 & 6 & 7 & 6 \end{bmatrix}$$

on the set N: 1, 2, 3, 4, 5, 6, 7, 8 are 2, 3, 4, 5, 6, 7. The elements 1 and 8 are not involved. Notice that we omitted 8 from the parenthesized expression. We may also omit the columns headed by 1 and 3, provided it is clear from the context that the set which the correspondence operates on contains them.

E5. Degree of a subset of N. The degree of the involved set of illustration E4 is 6, while the degree of the set not involved in the correspondence is 2.

E6. Equivalent correspondences.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 3 \end{bmatrix} \cong \begin{bmatrix} 3 & 4 & 2 & 1 \\ 4 & 3 & 2 & 2 \end{bmatrix}$$

while

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 3 \end{bmatrix} \ncong \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 3 \end{bmatrix}.$$

E7. Product of correspondences. Whenever correspondences are written in the parenthesized form, as found in this illustration, the bracket notation describing products may be avoided and the following scheme which determines a correspondence equivalent to one obtained by the first notation may be employed.

Let

$$M \cong \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ c_1 & c_2 & \cdots & c_n \end{bmatrix}.$$

Then

$$M \cong \left[\left\langle \begin{matrix} a_1 & b_1 \\ b_1 & d_1 \end{matrix} \right\rangle \left\langle \begin{matrix} a_2 & b_2 \\ b_2 & d_2 \end{matrix} \right\rangle \cdots \left\langle \begin{matrix} a_n & b_n \\ b_n & d_n \end{matrix} \right\rangle \right],$$

where the first column of

$$\left\langle \begin{matrix} a_i & b_i \\ b_i & d_i \end{matrix} \right\rangle$$

is obtained from the i-th column of the left-hand correspondence and the second column is that column from the right-hand correspondence which is headed by b_i, (the set of d's is the set of c's in some order), and

$$\left\langle \begin{matrix} a_i & b_i \\ b_i & d_i \end{matrix} \right\rangle$$

is to be interpreted as : write d_i under a_i. Then

$$M \cong \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ d_1 & d_2 & \dots & d_n \end{bmatrix}.$$

To be more particular:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \cong \left[\left\langle \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \right\rangle \left\langle \begin{matrix} 2 & 2 \\ 2 & 1 \end{matrix} \right\rangle \left\langle \begin{matrix} 3 & 3 \\ 3 & 1 \end{matrix} \right\rangle \right] \cong \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$

E8. C^n. Let C be equivalent to the correspondence of illustration E3. Then

$$C^4 \cong \begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ C_5 & C_6 & C_3 & C_4 & C_5 & C_6 \end{bmatrix}.$$

The reader may verify that if C is a correspondence on N, then the degree of the set involved in C^t is equal to or less than that of the set involved in C.

E9. Semi-group of correspondences. Let

$$C_1 \cong \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad C_2 \cong \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

then since $C_iC_j \in S$ for $i = 1, 2$ and $j = 1, 2$, this set of two C's forms a semi-group. The reader may verify that the totality of correspondences on a finite set N forms a semi-group. In particular if

$$1 \cong \begin{bmatrix} a & b \\ a & a \end{bmatrix}, \; 2 \cong \begin{bmatrix} a & b \\ b & b \end{bmatrix}, \; 3 \cong \begin{bmatrix} a & b \\ a & b \end{bmatrix}, \text{ and } 4 \cong \begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

a multiplication table for this semi-group is Table 4.1 at the end of this paper.

Our interpretation of the diagram of illustration E3 suggests to us a generalization of the concept of cycle of substitution group theory. Notice that the operator M acts as a cycle on the set: C_3, C_4, C_5, C_6 ; hence we may think of M as acting on these elements as a cycle and also as acting on the elements of the appendage (those written along the line joining the circle at C_3). Suppose that we consider a diagram, as for example the one in E10, consisting of a set of C's written along a circle, possibly a point circle with only one C on it, which circle has zero or more appendages each of which joins the circle at a C and has C's written on it, and which appendages each have zero or more subappendages each joining a previous appendage at a C; and these subappendages each have C's written on them, and possibly themselves have subappendages, etc., etc. But there must be at most a finite set of C's; and finally there may be a set of point circles different from the first circle mentioned each of which has exactly one C on it. Now as in the simpler case, let an operator move each C on an appendage of the appendaged circle to a C nearer this circle and along the same appendage line containing the first C, and move each C on the appendaged circle in clockwise motion to the next C on this circle (or into itself in case it is a point circle), and finally each C of all other circles into itself. The reader may draw such a diagram for the correspondence:

$$\begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} \\ C_2 & C_3 & C_2 & C_3 & C_3 & C_5 & C_5 & C_7 & C_7 & C_{10} \end{bmatrix}.$$

Such a correspondence is called an *excycle.*

To formalize; An excycle is a correspondence E on a set N: $A \cup B \cup C$, such that

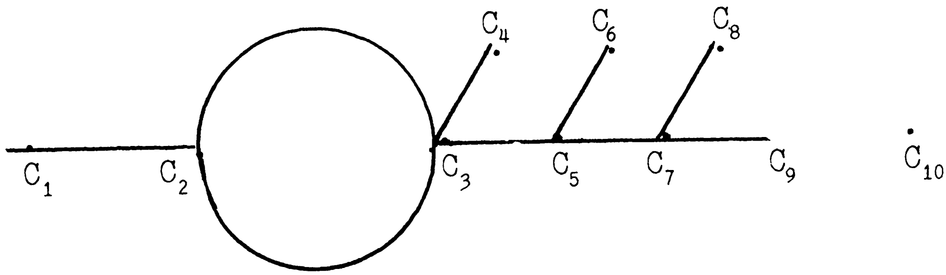
(i) If $n \in N$, $n \notin A$ and $a \in A$ then for each natural number i, $E^i[n] \notin A$, but $E[a] = a$.

(ii) If $c \in C$ and $d \in C$, there exists a natural number j such that $E^j[c] = d$. If C contains exactly one element, then B contains at least one element.

(iii) If $b \in B$, then for each natural number j, $E^j[b] \neq b$, but there exists a natural number k such that $E^k[b] \in C$.

Observe that A, B, and C are disjoint. If B is the null set, then E is a *cycle* or *cyclic substitution*. A cycle for which C contains exactly two elements is a *transposition*. If B and C each contain exactly one element, then E is a *repetition*. A *couplet* is either a repetition or a transposition. Let $b_1 \in B$ such that for each $b \in B$, $E[b_2] \neq b$ Then the set of elements of B of the type $E^k[b_1]$ together with b_1 is called a spur of E, and we may emphasize the role of b_1 by stating that b_1 determines this spur. If E contains at most r spurs, it is called an r-cycle. C is the *core* of E; B is the *exterior* of E; $B \cup C$ satisfies the definition already given of the set involved in E; and A is the set not involved in E.

E10. Excycles and subsets of the set an excycle operates on. Consider the diagram representing the excycle introduced preliminary to the formal definition of excycle:



Call the excycle E, and let N be the set of C's. The the elements of the non-involved set A, the exterior B, and the core C are respectively: C_{10}; C_1, C_4, C_6, C_8, C_9, C_7, C_5; and C_2, C_3. Notice that (i), (ii) and (iii) are each satisfied. The totality of spur sets, with elements: C_1; C_9, C_7, C_5; C_8, C_7, C_5; C_6, C_5; C_4 have as determing elements C_1; C_9; C_8; C_6;; and C_4 respectively. Notice that two spur sets may have either no elements in common or several elements in common. E is a 5-cycle.

Another example of excycle: The reader may verify that congruence modulo m divides the integers into m disjoint classes such that all the elements in a class are congruent, but no elements in different classes are congruent; suppose that we denote the classes by C_i, where C_i contains the integer i for $i = 1, 2, \ldots, m$; and that we define $C_i\ C_j$ to be congruent modulo m to the residue class C_k where k is the least positive residue of $i\ j$ modulo m; then if right-hand multiplication by C_3 is the operator and congruence modulo 9 the relation, a a correspondence may be written:

$$\begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \\ C_3 & C_6 & C_9 & C_3 & C_6 & C_9 & C_3 & C_6 & C_9 \end{bmatrix}$$

and is an excycle.

E11. Couplets. Lets the elements of N be 1, 2, 3, and let

$$C_1 \cong \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad C_2 \cong \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Then C_1 is a repetition and C_2 is a transposition. Each is a couplet. If E is the excyle of the second example of E10, notice that

$$E \cong \begin{bmatrix} C_3 \\ C_9 \end{bmatrix} \begin{bmatrix} C_7 \\ C_3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} \begin{bmatrix} C_4 \\ C_3 \end{bmatrix} \begin{bmatrix} C_6 \\ C_9 \end{bmatrix} \begin{bmatrix} C_2 \\ C_6 \end{bmatrix} \begin{bmatrix} C_5 \\ C_6 \end{bmatrix} \begin{bmatrix} C_8 \\ C_6 \end{bmatrix}.$$

PROBLEMS

In each of the first five problems, E is an excycle on $A \cup B \cup C$, where B and C are the exterior and core of E, respectively.

1. If $B = O$, $C = O$ and G is a correspondence on N, then $GE \cong EG \cong G$; here O stands for the null set. (Such and E is called an *identity* excycle.)

2. If $b \in B$ and t is a natural number, then $E^t[b] \in B \cup C$.

3. If $c \in C$ and t is a natural number, then $E^t[c] \in C$.

4. If C is of degree r, $c \in C$, and k is the smallest natural number such that $E^k[c] = c$. then k is r and the elements of C are the distinct elements: c, $E[c]$, ..., $E^{k-1}[c]$.

5. If C is of degree s and the spur of E of maximal degree is of degree r, then E generates a semi-group of correspondences whose non-equivalent elements are E, E, ..., E^{r+s-1}

6. If G is a correspondence on N, then G is either an excycle or is equivalent to a product of excycles on disjoint sets.

7. Under the hypothesis of problem 6, G is equivalent to a product of couplets.

8. If S is a semi-group of correspondences on N, it satisfies postulate 3 of the next section.

9. Construct a semi-group S of correspondences having the property:

   a. S contains an element E having the properties assigned to e in postulate 4 below.

   b. S satisfies postulate 5 below.

   c. S satisfies postulate 6 below.

d. If $A \in S$ and $B \in S$, then $AB \cong BA$.

10. Let N be an infinite set. Generalize the above definitions and work the above problems with infinite N and with added hypotheses where necessary.

Abstract algebraic systems of single composition.

We have had a double motive in our above discussion about correspondences: first, we wished to introduce the reader to a type of algebraic system which has its own intrinsic value, and to give him a set of more or less simple problems to acquaint him with some of the implications of the definitions; and second, we wished to be able to extract certain properties from semi-groups of correspondences and to use them to define some "abstract" algebraic systems. We shall proceed with the latter objective.

We begin with a non-null set S of elements. These elements may be denoted by latin letters. An operation o and an equivalence relation $\cong$ are related to S.

Consider a finite sequence (called a finite linearly ordered set in (**I**)) having the properties:

(i) The first element is an element of S.

(ii) The subsequent elements, if any, are alternately, respectively, the operation o and elements of S.

(iii) The last element is an element of S.

Such a sequence will be called a *combination on* S.

A *subcombination* of a given combination C is a combination which is either an element found in C or such an element followed by others in order as they appear in C.

The set S will satisfy some or all of the following postulates:

Postulate 1. If $a \in S$ and $b \in S$, then there exists an element c of S such that $aob \cong c$.

Postulate 2. If A is a combination, then $A \cong A$.

Postulate 3. If A, B, C, and D are combinations such that $A \cong B$ and $D \cong C$, with C a subcombination of B, and if B' is the combination obtained from B by replacing C by D, then $B' \cong A$.

Postulate 4. There exists an $e \in S$ such that $eoa \cong aoe \cong a$, for each $a \in S$.

Postulate 5. If a, b, c, and d are elements of S, and if $coa \cong cob$, then $a \cong b$; if $aod \cong bod$, then $a \cong b$.

Postulate 6. If a and b are elements of S, then there exist elements x and y of S such that $aox \cong b$ and $yoa \cong b$.

Postulate 7. We now introduce a new symbol of relation "$\not\cong$" and postulate that if A and B are combinations, then either $A \cong B$ or $A \not\cong B$, these relations being mutually exclusive.

If the set S satisfies postulates 1, 2, 3, and 7, it is called a *semi-group*. If it satisfies 1, 2, 3, 4, and 7, it is called a *gruppoid*. If it satisfies 1, 2, 3, 5, and 7, it is called a *skew-group*. If it satisfies 1, 2, 3, 6, and 7, it is called a *group*. The reader will recall from the discussion and problems above that each semi-group of correspondences satisfies the semi-group postulates; and that special semi-groups of correspondences satisfy the other definitions given above.

We have made no provision for the use of any parentheses in connection with combinations. Postulates 2 and 3 contain what is equivalent to the associative law as ordinarily employed. There seems to be no necessity for the use of parentheses in an associative system of single composition.

In applying our definition of abstract semi-groups, the meanings of the product symbol and the equivalence symbol are to be interpreted according to the particular type of semi-group which is being treated. Thus, for example, what is regarded as constituting equality in elementary arithmetic might be entirely different from what constitutes equivalence in connection with geometrical concepts.

Suppose that we have combinations M and N such that $M \cong N$. By postulate 2 we also have $N \cong N$.. Employing postulate 3, and noting that N by definition is a subcombination of itself, we obtain $N \cong M$. Hence, we have:

THEOREM 1 (Symmetry). If M and N are combinations, and $M \cong N$, then $N \cong M$.

THEOREM 2 (Transitivity). If $A \cong B$, and $B \cong C$, then $A \cong C$.

THEOREM 3 (Composition). If $A \cong B$, then for any combination C, $CoA \cong CoB$ and $AoC \cong BoC$.

It also follows by the substitution principle applied to $AoBoC \cong AoBoC$ that if $AoB \cong D$, and $BoC \cong G$, then $DoC \cong AoG$. This is the ordinary associative law.

Adopting the usual exponent notation, we let a^2 denote the combination aoa, etc. The *order* of an element a of a semi-group is defined as the maximum number of non-equivalent combinations contained in the set a, a^2, a^3, ..., provided such a number exists; if there are infinitely many non-equivalent combinations in the set, a is said to be of infinite order; if not, a is said to have finite order. An element of order one is called an *idempotent*. If an element a has finite order, then there is a least positive integer s such that $a^s \cong a^k$, where $0 < k < s$, and the set $a, a^2, \ldots, a^{s-1}$ is said to form a *cyclic semi-group* of order $s - 1$. It is said to be generated by a. The reader may show that this set is a semi-group. If the semi-group generated is a group, it is called a *cyclic group*. The integer k defined above is necessarily unique. The *period* of a is defined as $s - k$; we note that the order of a is $s - 1$.

An element e of a semi-group S is said to be a *left (right) identity* provided that if $a \in S$, then $eoa \cong a (aoe \cong a)$. An element which is both a right and a left identity is called and *identity* element.

An element h of a semi-group S is said to be a *left (right) annihilator* (or *annulator*) provided that if $a \in S$, then $hoa \cong h (aoh \cong h)$. An element which is both a right and a left annihilator is said to be an *annihilator*.

If S is a semi-group, we define a *subsemi-group* of S as a subset of S which is itself a semi-group with the same operation and equivalence as S. It follows that the set of left identities, if any, of a semi-group form a subsemi-group of the given semi-group. In this subsemi-group, each element is a left identity and a right annihilator, for if e_1 and e_2 are two such left identities, then $e_1 o e_2 \cong e_2$

An element a of a semi-group S is said to be *left (right) cancellable* if $a, b \in S$ and whenever $aob \cong aoc (boa \cong coa)$, then $b \cong c$. A *cancellable* element is one which is both right and left cancellable. An element a of a semi-group is said to be *left non-cancellable* if there exist non-equivalent elements b and c of the semi-group such that $aob \cong aoc$. A similar definition holds for *right non-cancellable*.

Problems

11. If a semi-group has two non-equivalent left identities, then it has no right identities. If S is a skew-group with $aob \cong b$, $a \in S$, and $b \in S$, then a is an identity.

12. If a semi-group has identities e_1 and e_2, then $e_1 \cong e_2$; also, if it has annihilators a_1 and a_2, then $a_1 \cong a_2$.

13. If the intersection of two subsemi-groups of a semi-group S is non-null, then this intersection is a subsemi-group of S.

14. If a semi-group S contains a cancellable element, the set of such forms a subsemi-group of S.

15. If a' and a are elements of a semi-group and a' is left non-cancellable, then aoa' is left non-cancellable.

16. If a semi-group S contains a cancellable element of finite order, then S is a gruppoid.

17. If a skew-group contains a subgruppoid, then the skew-group

itself is a gruppoid, with the same identity element as the subgruppoid.

18. Each finite skew-group is a group.

19. If a generates a cyclic group of order s and k is prime to s, then a^k generates this cyclic group also.

Consider a semi-group S with operation o and equivalence $\cong$, such that if $a \cong b$, then a and b are the same element of S; consider also a semi-group S' with operation o' and equivalence $\cong'$, such that if $c \cong' d'$, then c' and d' are the same element of S'. A mapping M of S onto S' is called a *homomorphism* of S onto S' provided that for $a \in S$, $b \in S$, $a' \in S'$, and $b' \in S'$, such that $M[a] = a'$ and $M[b] = b'$, it follows that $M[aob] = a'o'b'$. If such a mapping exists, S is said to be homomorphic to S'.

If $a \in S$, $b \in S$ and $a' \in S'$, $b' \in S'$ and M is a homomorphism of S onto S' such that $M[a] = a'$ and $M[b] = b'$, and if it follows from $a' \cong' b'$ that $a \cong b$, then M is an *isomorphism of S onto S'*; again if such a mapping exists, S is said to be isomorphic to S'.

A homomorphism of a semi-group S onto a subset S' of S is an *endomorphism on S*, while an isomorphism of a semi-group S onto itself is an *automorphism on S*. The set C of elements of a semi-group S is the *central* (or *center*) of S provided it is true that if $c \in C$ and $s \in S$ then $cos \cong soc$. If C is non-null, then C is a semi-group. A semi-group S with central C is said to be *Abelian* if $C = S$.

A semi-group S is said to be *embedded* in a semi-group T if T contains a subsemi-group S' isomorphic to S.

E12. Homomorphism. Let I be the semi-group of natural numbers with operation multiplication and ordinary equality as its equivalence relation, and S be the semi-group with operation, multiplication and ordinary equality as its equivalence relation, containing only 0. Let M be a mapping such that $M[i] = 0$ for each $i \in E$. Then since M is an onto mapping and $M[i_1 i_2] = 00$ for each i_1, $i_2 \in I$, M is a homomorphism.

E13. Isomorphism. Let I be the semi-group of positive integers and $-I$ be the semi-group of negative integers, each of which has addition as its operation and ordinary equality as its equivalence relation. Let N be a mapping such that $N[i] = -i$ for each $i \in I$; then since N is a mapping of I onto $-I$ and

$$N[i_1 + i_2] = (-i_1) + (-i_2)$$

for each i_1, $i_2 \in I$ and since $-i \cong -j$ entails $i \cong j$ for each $-i$, $-j \in -I$, I is isomorphic to $-I$.

E14. Endomorphism. Let R be the semi-group of integers with operation, addition and ordinary equality as the equivalence relation, and let S be the set whose elements are: 1, 2, 3, ..., m, $m > 1$.

The reader may show that S forms a semi-group, with operation addition modulo m and with equivalence, congruence modulo m. It is known ((I), p. 12) that there exist integers k and t, $0 < t \leqq M$ such that if $i \in R$, $i = kn + t$. Let E be the mapping on R such that $E[mk + t] = t$, where $k \in R$ and $0 < t \leqq m$. Then since E is an "onto" mapping such that

$$E[mk + t] = t,$$

$$E[mk_1 + t_1] = t_1,$$

for $k_1 \in R$ and $0 < t_1 \leqq m$ entail

$$E[mk + t + mk_1 + t_1] = E[m(k + k_1) + t + t_1] = t + t_1,$$

E is an endomorphism of R onto S.

E15. Automorphism. Consider the cyclic group of order s generated by the element a; let k be prime to s, and A be a correspondence such that $A[a^i] = a^{ki}$ for $i = 1, 2, \ldots, s$; then if

$$A[a^{i_1}] = a^{ki_1}, \quad A[a^{i_2}] = a^{ki_2},$$

for $1 \leqq i_1 \leqq s$, and $1 \leqq i_2 \leqq s$, it follows that

$$A[a^{i_1}a^{i_2}] = A[a^{i_1 + i_2}] = a^{k(i_1 + i_2)},$$

and since each $a^{ki} \cong a^{kj}$ entails $a^i \cong a^j$ for $1 \leqq i \leqq s$ and $1 \leqq j \leqq s$, A is an automorphism.

E16. Center and Abelian semi-group. The reader may verify that the center of the semi-group of all correspondences on the set: 1, 2, contains only the identity correspondence. He may also verify that the residue classes modulo m form an Abelian semi-group under multiplication modulo m and congruence modulo m.

E17. The embedding of one semi-group in another. The semi-group of negative integers under addition and ordinary equality is imbedded in the semi-group of non-negative integers under addition and ordinary equality.

Problems

20. If a finite semi-group consists of powers of one element a and s is the least integer such that, for some k, $s > k > 0$, it is true that $a^k \cong a^s$, then the elements $a^k, a^{k+1}, \ldots, a^{s-1}$ form a cyclic group. Show how to determine the generators of this group.

21. Show that if n is a positive integer, then the elements $1 + xn/y$, where x ranges over the set of integers, and y ranges independently over the integers prime to n, form a group under multiplication.

Let H be a homomorphism of the semi-group R onto the semi-group S. We may denote the elements of R by small latin letters and their images in S by the primes of these latin letters; similarly we may denote subsets of R by capital letters and the corresponding subsets of S by the primes of these capital letters. These remarks apply to problems 22-28.

22. If R contains an idempotent, H maps it onto an idempotent.

23. If r is a right identity (left identity, identity) of R, r' is a right identity (left identity, identity) of S.

24. If a is a right annulator (left annulator, annulator) of R, a' is a right annulator (left annulator, annulator) of S.

25. If c has finite order, then the order of c' is equal to or less than that of c.

26. If U is a subset of R which is a semi-group, then U' is a sub-semi-group of S.

27. If the U of problem 26 is commutative, then so is U'.

28. If the U of problem 26 is cyclic, then so is U'.

29. Each finite semi-group is isomorphic to some semi-group of correspondences.

30. Suppose that each element of a semi-group S has finite order; then it follows from problem 20 that if $a \in S$, there exists a natural number t such that a^t is an idempotent i_a. Assume that S is commutative and let E be a correspondence on S such that if $a \in S$ then $E[a] = i_a$. Prove that E is an endomorphism.

31. Let S be a finite commutative semi-group which satisfies:
   (i) If $a \in S$ then a is an idempotent.
   (ii) There exists a subset S^* of S such that each element of S is expressible uniquely except for order as a product of non-equivalent elements of S^* and such that no element of S^* has a divisor other than itself.

Further let A be a substitution on S such that
   (i) If $p_i \in S^*$, then $A[p\ ] = p_i'$ where $p_i' \in S^*$.
   (ii) If $a \in S$ and

$$a \cong \prod_{n=1}^{r} p_n$$

is a factorization of a into a product of distinct elements of S^*, then

* "Representations of Finite Simple Semi-Groups," *Duke Math. Journal* 11, 251-265, 1941.

$$A\,[a] = \prod_{i=1}^{r} p_i{}' .$$

Prove that A is an automorphism, and conversely that each automorphism on S satisfies the above definition of A.

We desire a method of constructing semi-groups of a certain order. Let S be a semi-group whose elements are denoted by 1, 2, ..., n, and S' be the set whose elements are denoted by 1, 2, ..., $n+1$; further let S'' be the set: $K_1, K_2, \ldots, K_n$ of correspondences defined by:

(i) $K_i[n+1] = i$ for $i = 1, 2, \ldots, n$.

(ii) If $j,\ k \in S$ such that $jk \cong m$, then $K_k[j] = m$.

R. R. Stoll,[1] in proving problem 29 above, proved that the mapping I such that $I[i] = K_i$, for $i = 1, 2, \ldots, n$ is an isomorphism of S onto S''. For example, if S has as a multiplication table, 2.2 at the end of this paper, then

$$I\ [1] = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad I\ [2] = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix}.$$

Apparently, then, all the semi-groups of order n, (the phraseology, "all the semi-groups or order n" is defined in the next paragraph) are embedded in the semi-group of all correspondences on a set of degree $n+1$. When $n > 3$, since there are n^n correspondences on such a set, it is a lengthly job to construct all the semi-groups of order n using this method. A. R. Pool constructed all the commutative semi-groups of orders 2, 3, and 4. K. S. Carman, J. C. Harden, and E. E. Posey constructed *all* the semi-groups of orders 2, 3, and 4, and those of orders 2 and 3 are reproduced below. T. S. Motzkin and J.L. Selfridge, in the early summer of 1955, constructed all the semi-groups of order 5.

Two semi-groups R and S are said to have the property D if they satisfy either of the conditions:

(i) R is not isomorphic to S.

(ii) No one to one mapping M of R onto S exists such that as x and y vary independently over R, if $M[x] = x'$ and $M[y] = y'$, then $M[xy] = y'x'$. (Such a mapping is called a skew-isomorphism.)

Each pair of semi-groups whose multiplication tables are given below has the property D. The semi-groups of order two whose tables appear below is a maximal set of semi-groups of order two, each two of which have the property D, and likewise for the tables for semi-groups of order three. In this connection, by "all the semi-groups of order n", we mean any maximal set of semi-groups of order n, each two semi-groups of which have the property D. Table 4.1 is given to illustrate the table for the last semi-group of E9. Each type which is not otherwise illustrated, of finite semi-group mentioned in either examples or problems in this paper, is exemplified by a semi-group which has a multiplication table below. Let S of order i have a table below. Its elements are denoted by 1, 2, ..., i. If $r, s \in S$, the product rs is found in the r-th row and s-th column of the table.

SEMI-GROUPS OF ORDER TWO

2.1	2.2	2.3	2.4
1 2	1 1	1 1	1 1
2 1	1 2	2 2	1 1

SEMI-GROUPS OF ORDER THREE

3.1	3.2	3.3	3.4	3.5	3.6
1 2 2	1 1 1	1 2 3	1 1 3	1 1 3	1 1 1
2 1 1	1 1 1	2 3 1	1 1 3	1 2 3	1 1 1
2 1 1	1 1 2	3 1 2	3 3 1	3 3 1	1 1 3

3.7	3.8	3.9	3.10	3.11	3.12
1 1 1	1 1 1	1 1 1	1 1 3	1 2 3	1 1 1
1 2 1	1 1 2	2 2 2	1 1 3	2 1 3	2 2 2
1 3 1	1 2 3	1 1 1	3 3 3	3 3 3	1 1 3

3.13	3.14	3.15	3.16	3.17	3.18
1 1 1	1 1 1	1 1 3	1 1 1	1 1 1	1 1 1
2 2 2	2 2 2	2 2 3	1 2 2	1 2 1	1 1 1
1 2 3	3 3 3	3 3 3	1 2 3	1 1 3	1 1 1

A SEMI-GROUP OF ORDER FOUR

4.1
1 2 1 2
1 2 2 1
1 2 3 4
1 2 4 3

The University of Texas

MISCELLANEOUS NOTES

Edited by

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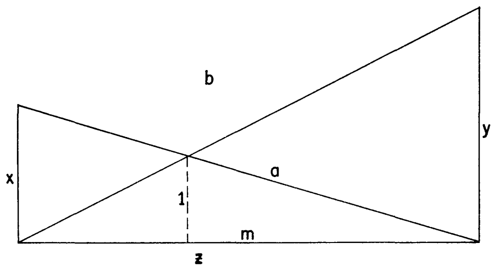
THE CROSSED LADDERS

H. A. Arnold

In the old Mathematics News Letter [1] this problem was resolved by solving a quartic equation with all coefficients present. The present note considers a more general situation, where preliminary substitutions lead to a quartic with a quick iterative solution that has been of real illustrative value in courses in numerical analysis.

A ladder of length a connects the base of a vertical wall, W_a, to a point at height x on a parallel wall W_b. Similarly, a ladder of length b rests at the base of W_b and on a point of W_a, y units high. The walls are z units apart. They cross at height unity and distance m from W_a. The knowns are a and b, usually taken as 3 and 4 respectively The unknowns are x, y, m and z. Usually only z is required.



By similar triangles and the Pythagorean theorem

$$y + x = xy, \text{ from } m/z = (y - 1)/y = 1/x = 1 - 1/y; \qquad (1)$$

$$y^2 - x^2 = b^2 - a^2 = K, \text{ from } z^2 = b^2 - y^2 = a^2 - x^2 \qquad (2)$$

In (1) and (2), set

$$y + x = u, \; y - x = v; \; u + v = 2y, \; u - v = 2x; \qquad (3)$$

$$uv = K \quad u^2 - v^2 = 4xy = 4(x + y) = 4u. \qquad (4)$$

Eliminating v, $f(u) = u^4 - 4u^3 - K^2 = 0$, or

$$u = 4 + K^2/u^3 \qquad (5)$$

In case $b = 4$, and $a = 3$, $K = b^2 - a^2 = 7$, there is exactly one positive root, u, and $u > 4.5$, as is easily seen from a graph. It may be approximated to by the very rapid and simple iterative formula

$$u_{n+1} = 4 + 49/u_n^3 \quad (6)$$

To five decimals

$$u = 4.52786, \quad v = 1.54598, \quad z = 2.60329.$$

With general K, if following (5), we write

$$u_{n+1} = \phi(u_n), \quad (7)$$

then a condition for convergence of the iteration is $|\phi'| < 1$ in the neighborhood of the root, [2]. Here, this becomes $u > |\sqrt[4]{3K^2}$, and if $K^2 < (16)(27)$, on substitution, $f(\sqrt[4]{3K^2}) < 0$ and $f(+\infty) > 0$, so a positive root will be assured, calculated by (7). There will also be a negative root and two conjugate imaginary roots of $f(u) = 0$, by a quick examination of the graph of $y = f(u)$.

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(1) Problem No. 35. Mathematics News Letter, 8, 65-68 (1933).

(2) J. B. Scarborough, Numerical Mathematical Analysis, Johns Hopkins Press Baltimore, 1950, p.201.

A VECTORIAL DERIVATION OF CRAMER'S RULE

W. J. Klimozak, Trinity College

The purpose of this paper is to present a vectorial method of solving an independent system of linear equations of the form

$$(1) \qquad \sum_{k=1}^{n} \alpha_{j,k} x_k = \beta_j, \quad \beta_j \neq 0 \qquad (j = 1, 2, \ldots, n),$$

which reduces to Cramer's rule. Let $a_{j,k} = \alpha_{j,k}/\beta_j$ and define the n fixed vectors,

$$A_j = (a_{j,1}, a_{j,2}, \ldots, a_{j,n}) \quad (j = 1, 2, \ldots, n),$$

and the unknown vector,

$$x = (x_1, x_2, \ldots, x_n),$$

Then the system (1) may be written in the form

$$(2) \qquad A_j \cdot x = 1 \qquad (j = 1, 2, \ldots, n),$$

where $A_j \cdot x$ denotes the scalar product of A_j and x. The method consists of determining the direction and magnitude of x in terms of the known vectors A_j $(j = 1, 2, \ldots, n)$.

We shall first develop the solution of (1) then $n = 3$ and then extend the results to the general case. For this purpose it will be convenient to consider any vector x as a directed line segment from the origin to the point having rectangular coordinates x_1, x_2, and x_3. If $|A_j|$ and $|x|$ denote the magnitudes of A_j and x respectively and $\angle(A_j, x)$ the angle between A_j and x then from the definition of scalar product we may write (2) as

$$(3) \qquad |A_j||x| \cos \angle(A_j, x) = 1 \qquad (j = 1, 2, 3).$$

It then follows that each of the angles $\angle(A_j, x)(j = 1, 2, 3)$ is acute. Also since $|x| \neq 0$, we have

$$(4) \qquad |A_1| \cos \angle(A_1, x) = |A_2| \cos \angle(A_2, x) = |A_3| \cos \angle(A_3, x).$$

This shows that the direction of x is such that the projections of the vectors $A_j(j = 1, 2, 3)$ on x are all equal. Hence x is directed along the normal line to the plane $P(A_1, A_2, A_3)$ determined by the terminal ends of the vectors $A_j(j = 1, 2, 3)$ so that (A_1, x) is acute. Let $v = e(A_2 - A_1) \times (A_3 - A_1)$, where $(A_2 - A_1) \times (A_3 - A_1)$ denotes the vector product of $A_2 - A_1$ and $A_3 - A_1$, and e is +1 or -1 so that $\angle(v, x)$ will be acute. Then x has the same direction as v.

To obtain the magnitude of x, we have from (3)

$$|x| = 1/[|A_1| \cos \angle(A_1, x)].$$

Thus $|x|$ is the inverse of the distance between the origin and the plane $P(A_1, A_2, A_3)$. This distance is the component of A_1 in the direction of v, so that

$$|x| = |(A_2 - A_1) \times (A_3 - A_1)|/|(A_1 \cdot (A_2 - A_1) \times (A_3 - A_1)|.$$

Since $x = |x|v/|v|$ and e is +1 or -1 according as $A_1 \cdot (A_2 - A_1) \times (A_3 - A_1)$ is positive or negative,

$$(5) \qquad x = (A_2 - A_1) \times (A_3 - A_1)/[A_1 \cdot (A_2 - A_1) \times (A_3 - A_1)]$$

as the vector form of the solition of (1).

Using the determinant notation for the vector product, we may write

$$(6) \quad (A_2 - A_1) \times (A_3 - A_1) = \begin{vmatrix} i_1 & i_2 & i_3 \\ a_{2,1} - a_{1,1} & a_{2,2} - a_{1,2} & a_{2,3} - a_{1,3} \\ a_{3,1} - a_{1,1} & a_{3,2} - a_{1,2} & a_{3,3} - a_{1,3} \end{vmatrix},$$

where i_1, i_2, i_3 are the positive unit vectors along the x_1, x_2, x_3 axes respectively. Also

$$(7)\quad A_1\cdot(A_2-A_1)\times(A_3-A_1) = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1}-a_{1,1} & a_{2,2}-a_{1,2} & a_{2,3}-a_{1,3} \\ a_{3,1}-a_{1,1} & a_{3,2}-a_{1,2} & a_{3,3}-a_{1,3} \end{vmatrix}.$$

By successively adding the first row of the determinant in (7) to the second and third rows, we see that $A_1\cdot(A_2 - A_1) \times (A_3 - A_1)$ is the determinant $\Delta(a_{1,1},\ a_{2,2},\ a_{3,3})$ of the coefficients of (2)when $n = 3$, From (6) we find that the i_1 component of the vector $(A_2 - A_1) \times (A_3 - A_1)$ is

$$\begin{vmatrix} a_{2,2}-a_{1,2} & a_{2,3}-a_{1,3} \\ a_{3,2}-a_{1,2} & a_{3,3}-a_{1,3} \end{vmatrix} = \begin{vmatrix} 1 & a_{1,2} & a_{1,3} \\ 0 & a_{2,2}-a_{1,2} & a_{2,3}-a_{1,3} \\ 0 & a_{3,2}-a_{1,2} & a_{3,3}-a_{1,3} \end{vmatrix}.$$

By successively adding the first row of the last determinant to the second and third rows, we obtain as the i_1 component of $(A_2 - A_1) \times (A_3 - A_1)$ the determinant which results when each entry of the first column of $\Delta(a_{1,1},\ a_{2,2},\ a_{3,3})$ is replaced by 1. We shall substitute c_j for $a_{j,j}$ in the notation $\Delta(a_{1,1},\ a_{2,2},\ a_{3,3})$ to denote the determinant obtained from $\Delta(a_{1,1},\ a_{2,2},\ a_{3,3})$ by replacing the entries in its j-th column by c_1, c_2, c_3 respectively. Then the i_1 component of $(A_2 - A_1) \times (A_3 - A_1)$ is $\Delta(1,\ a_{2,2},\ a_{3,3})$. Similarly the i_1 and i_3 components of $(A_2 - A_1) \times (A_3 - A_1)$ are the determinants $\Delta(a_{1,1},\ 1,\ a_{3,3})$ and $\Delta\ (a_{1,1},\ a_{2,2},\ 1)$ respectively. If we let $\beta = \beta_1\beta_2\beta_3$, we note that $\Delta(\alpha_{1,1}\ \ \alpha_{2,2}\ \ \alpha_{3,3}) = \beta\Delta(a_{1,1},\ a_{2,2},\ a_{3,3})$, $\Delta(\beta_1,\ \alpha_{2,2},\ \alpha_{3,3}) = \beta\,\Delta(1,\ a_{2,2},\ a_{3,3})$, $\Delta\,(\alpha_{1,1},\ \beta_2,\ \alpha_{3,3}) = \beta\,\Delta\,(a_{1,1},\ 1,\ a_{3,3})$, and $\Delta\,(\alpha_{1,1},\ \alpha_{2,2},\ \beta_3) = \beta\,\Delta\,(a_{1,1},\ a_{2,2},\ 1)$. Hence we may write (5) as

$$x_1i_1 + x_2i_2 + x_3i_3 = [\Delta(\beta_1,\ \alpha_{2,2},\ \alpha_{3,3})i_1 + \Delta(\alpha_{1,1},\ \beta_2,\ \alpha_{3,3})i_2 + \Delta(\alpha_{1,1},\ \alpha_{2,2},\ \beta_3)i_3]/\Delta(\alpha_{1,1},\ \alpha_{2,2},\ \alpha_{3,3}),$$

which is the statement of Cramer's rule in vector form applied to (1) for $n = 3$.

All of the results obtained for $n=3$ have n-dimensional analogues. In the general case the direction of x is orthogonal to the hyperplane

$P(A_1, A_2, \ldots, A_n)$ determined by the terminal ends of the vectors A_j ($(j = 1, 2, \ldots, n)$ and such that $\angle(A_1, x)$ is acute. This direction is the same as the direction of the vector v defined as follows:

$$v = e\prod_{j=2}^{n}(A_j - A_1) = e\begin{vmatrix} i_1 & & i_2 & & \cdots & i_n & \\ a_{2,1} & -a_{1,1} & a_{2,2} & -a_{1,2} & \cdots & a_{2,n} & -a_{1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & -a_{1,1} & a_{n,2} & -a_{1,2} & \cdots & a_{n,n} & -a_{1,n} \end{vmatrix},$$

where i_k $(k = 1, 2, \ldots, n)$ are the positive unit vectors along the x_k $(k = 1, 2, \ldots, n)$ axes respectively, and e is +1 or -1 so that $\angle(v,x)$ will be acute. Corresponding to (5) we have

$$x = \prod_{j=2}^{n}(A_j - A_1)/[A_1 \cdot \prod_{j=2}^{n}(A_j - A_1)],$$

which may be written as

$$\sum_{k=1}^{n} x_k i_k =$$

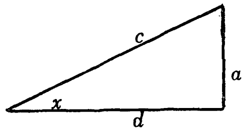
$$\sum_{k=1}^{n} \Delta(\alpha_{1,1}, \ldots, \alpha_{k-1,k-1}, \beta_k, \alpha_{k+1,k+1}, \ldots, \alpha_{n,n}) i_k / \Delta(\alpha_{1,1}, \ldots \alpha_{n,n}).$$

This is the vector form of the solution of (1) by Cramer's rule.

A "Reference Triangle" for Hyperbolic Functions

Dr. W. Van Voorhis

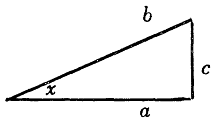
When the value of any one trigonometric function is given, it is often convenient to construct and label a "reference right triangle" from which the numerical values of other trigonometric functions may be read. The proper sign for a specific function may be prefixed according to the size of the angle. For example, if $\sin x = -(a/c)$ where a and c are positive, and x when placed in standard position terminates in the third quadrant, the values of the other trigonometric functions may be determined from the following right triangle



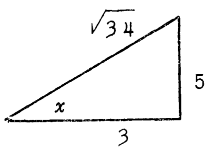
in which b has been calculated from a and c, and use of the proper sign. That is, we obtain cos $x = -(b/c)$, tan $x = a/b$, and so forth.

The reference right triangle for trigonometric functions is a familiar device, and its use presents little or no difficulty in operations with these functions. In fact, it is probably more economical than to resort to the use of appropriate identities. A difficulty does arise, however, when the student begins the study of hyperbolic functions and attempts, erroneously, to make use of the same device.

Although the hyperbolic functions are in no way related to the circular functions, it is possible to construct a simple "reference triangle" from which the numerical values of hyperbolic functions may be read when the value (other than zero) of any one hyperbolic function is given. This "trick" device, the construction of which is possible because of certain fundamental identities involving hyperbolic functions, is obtained by *rotating* the letters on the trigonometric reference triangle one position in the clockwise direction. That is, the "reference triangle" for hyperbolic functions becomes:



On this device the numerical values for sinh x, cosh x, tanh x, etc., are *labeled* and *read* $a/c, b, c, a/b$, etc., as indicated, and the proper sign is prefixed according as x is positive or negative. For example, if sinh $x = -(3/5)$, the 3 and 5 are placed on the reference triangle in the following way:



and the value for the third side is calculated by use of the Pythagorean Theorem as in the case of trigonometric functions.

Since x is negative, we obtain cosh $x = \sqrt{34}/5$, tanh $x = -(3/\sqrt{34}$, etc. The rotation required to label and to read the device correctly is quickly learned by the student; and the device not only provides motivation, but is quite useful for rapidly calculating the values of hyperbolic functions appearing in expressions to be evaluated.

Penn College

ONE SIDE TANGENTS

William R. Ransom

In plane geometry the tangent to a circle touches it at but one point and all its other points lie on one side of the curve. This property may also be used for deriving the equation of a tangent to a conic: the commonly used concept of a limiting position of a tangent is therefore not actually necessary.

Take the point (p,q) on the ellipse or hyperbola, $b^2x^2 \pm a^2y^2 = a^2b^2$, and consider the tangent at that point: its equation is $y = q + M(x-p)$, where M is to be determined so that the line does not cross the curve. Let us take the two points (x,C) on the curve and (x,T) on the tangent. According to the one side definition of the tangent $C-T$ must have the same sign on both sides of (p,q). Transforming, we have

$$C-T = C-q-M(x-p) = \frac{C^2-q^2}{C+q} - M(x-p)$$

$$= \pm\frac{b^2}{a^2}\,\frac{p^2-x^2}{C+q} - M(x-p) = (x-p)\left[\mp\frac{b^2}{a^2}\,\frac{x+p}{C+q} - M\right]$$

As (x,C) passes through (p,q) the factor $(x-p)$ changes sign, and as $C-T$ does not change sign, the other factor must change sign. So it is necessary that M have the value $(\mp b^2/a^2)(p+p)/(q+q)$. It is also sufficient that it have this value, for then the factor in the brackets is

$$\mp\frac{b^2}{a^2}\,\frac{x+p}{C+q} - \frac{b^2p}{a^2q} = \mp\frac{b^2}{a^2}\,\frac{xq+pq-pC-pq}{q(C+q)} = \mp\frac{b^2}{a^2}\,\frac{q(x-p)+p(q-C)}{q(C+q)}$$

Both $(x-p)$ and $(q-C)$ change sign as (x,C) passes through (p,q), hence this factor changes sign there, and $C-T$ has the same sign on both sides of (p,q).

With this value of M, the equation for the tangent becomes

$y = q \mp (b^2p/a^2q)(x-p)$ which reduces to $b^2px \pm a^2qy = b^2p^2 \pm a^2q^2 = a^2b^2$.

For the parabola in the form $y = kx^2$, using the same notation we have $C-T = kx^2 - q - M(x-p) = kx^2 - kp^2 - M(x-p) = (x-p)[k(x+p) - M]$. Here we see that it is necessary that $M = k(p+p)$, and this is sufficient for it makes $C-T = k(x-p)^2$, which is positive on both sides of (p,q).

For the hyperbola in the form $xy = a^2$, we have

$$C - T = a^2/x - a^2/p - M(x - p) = (x - p)\,[- a^2/px - M]$$

and M must have the value $-a^2/p^2$, which is sufficient for it gives

$$C - T = a^2(x+ p)^2/p^2x.$$

The same method may be applied to many other curves. For example, take $y^2 = x^3$, the Neilian parabola. For $C - T$, this gives

$$x\sqrt{x} - p\sqrt{p} - M(x - p) = (\sqrt{x} - \sqrt{p})\,[x + \sqrt{px} + p - M(\sqrt{x} + \sqrt{p})]$$

For the factor in brackets to change sign at (p, q) we must have

$$M = (p + \sqrt{pp} + p)/(\sqrt{p} + \sqrt{p}) = 3\sqrt{p}/2$$

$C - T$ then reduces to

$$\tfrac{1}{2}(\sqrt{x} - \sqrt{p})[2x + 2\sqrt{px} + 2p - 3\sqrt{px} - 3p]$$

$$= \tfrac{1}{2}(\sqrt{x} - \sqrt{p})\,[2x - 2\sqrt{px} + \sqrt{px} - p]$$

$$= \tfrac{1}{2}(\sqrt{x} - \sqrt{p})[2\sqrt{x}(\sqrt{x} - \sqrt{p}) + \sqrt{p}(\sqrt{x} - \sqrt{p})]$$

$$= \tfrac{1}{2}(\sqrt{x} - \sqrt{p})^2\;(2\sqrt{x} + \sqrt{p})$$

which does not change sign at (p, q).

Note that in this argument x does not approach p as a **limit**: we take $x = p$ in determining M, and $x \neq p$ in proving that $C - T$ does not change sign.

Tufts College

A REPRESENTATION OF THE COMMUTATOR SUBGROUP

Peter Yff

Since the set of products $a^{-1}b^{-1}ab$ in a group G generates the commutator subgroup of G, one may be interested in the subgroup generated by products of the type $a^{-1}b^{-1}c^{-1}abc$ or even $a_1^{-1}\ldots a_n^{-1}a_1\ldots a_n$. A brief investigation yields the following theorem:

The set of all products $a_1^{-1}\ldots a_n^{-1}a_1\ldots a_n$ of elements a_i in a group G is the commutator subgroup of G.

Proof: (a) Since $a_1^{-1}\ldots a_n^{-1}a_1\ldots a_n$ is a commutator when $n=1$ or 2, we make the inductive assumption that it is a product of commutators when $n=k$.

Then $a^{-1}\ldots a_{k+1}^{-1}a_1\ldots a_{k+1}$

$$= a_1^{-1}\ldots a_k^{-1}a_1\ldots a_k(a_1\ldots a_k)^{-1}a_{k+1}^{-1}(a_1\ldots a_k)a_{k+1}.$$

The product $(a_1\ldots a_k)^{-1}a_{k+1}^{-1}(a_1\ldots a_k)a_{k+1}$ is a commutator, so by induction $a_1^{-1}\ldots a_n^{-1}a_1\ldots a_n$ is always a product of commutators.

(b) Now assume that any product of m commutators can be expressed in the form $a_1^{-1}\ldots a_r^{-1}a_1\ldots a_r$. This is obviously true when $m=1$.

Multiplying by another commutator, we obtain

$$a_1^{-1}\ldots a_r^{-1}a_1\ldots a_r s^{-1}t^{-1}st$$
$$= a_1^{-1}\ldots a_r^{-1}(ts)s^{-1}t^{-1}a_1\ldots a_r(ts)^{-1}st$$
$$= a_1^{-1}\ldots a_{r+3}^{-1}a_1\ldots a_{r+3},$$

where $a_{r+1}=(ts)^{-1}$, $a_{r+2}=s$, and $a_{r+3}=t$.

Therefore any product of commutators can be expressed in the form $a_1^{-1}\ldots a_n^{-1}a_1\ldots a_n$, and the proof is complete.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

257. *Proposed by C. S. Ogilvy, Hamilton College, New York.*

Given a curve $y = f(x)$ and a straight line $Ax + By + C = 0$ intersecting at P and Q so as to bound an area, and such that any normal to the line between P and Q meets the curve exactly once; find the volume of the solid formed by rotating the area about the line.

258. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

A triangle ABC inscribed in a circle varies such that AB and AC keep fixed directions. Find the locus of the orthocenter H.

259. *Proposed by N. Shklov, University of Saskatchewan.*

Let A and B be the feet of the perpendicular drawn from the variable point $P(x,y)$ to the lines $15x - 8y = 0$ and $y = 0$ respectively. If the length of $AB = 15$, what is the equation of the locus of P?

260. *Proposed by Ben K. Gold, Los Angeles City College.*

A student solved the following problem incorrectly. Problem: In how many ways can five dice be tossed so that at least three aces show? His solution was ${}_5C_3 6^2$, reasoning that three dice must be aces and the other two may or may not be. What is the fallacy in this reasoning and how can the correct solution be obtained from his incorrect answer?

261. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Determine the entire class of analytic functions $F(x)$ so that Simpson's Quadrature Formula

$$\int_{-h}^{h} F(x)\, dx = \frac{h}{3}\left[F(-h) + 4\,F(0) + F(h)\right]$$

holds exactly.

262. *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Partition 166,665 = 3(55,555) into the sum of three positive palindromes in eight different ways, with no zeros involved and with all 24 palindromes distinct.

263. *Proposed by Chih-yi Wang, University of Minnesota.*

Define $g(z) = [(|z| + z)/2]^2$, $g^2(z) = g(g(z))$, ..., $g^n(z) = g(g^{n-1}(z))$. Show that for $z = re^{i\alpha}$, $0 < \alpha \leq \pi$, $0 \leq r \cos^2(\alpha/2) < 1$ we have $\lim_{n\to\infty} g^n(re^{i\alpha}) = 0$.

SOLUTIONS

Late Solutions

223, 224, 225, 226, 227, *Richard K. Guy, University of Malaya, Singapore.*

230. *J. M. Gandhi, Lingraj College, Belgaum, India; T. F. Mulcrome, St. Charles College, Grand Coteau, Louisiana; M. S. Klamkin, Polytechnic Institute of Brooklyn.*

232. *J. M. Gandhi, Lingraj College, Belgaum, India.*

234. *M. S. Klamkin, Polytechnic Institute of Brooklyn.*

235. *J. M. Gandhi, Lingraj College, Belgaum, India; M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Binomial Coefficients

236. [May 1955] *Proposed by C. W. Trigg, Los Angeles City College.*

What is the largest value of y such that there is a binomial expansion in which the coefficients of y consecutive terms are in the ratio $1:2:3: \ldots :y$? Identify the corresponding expansion and the terms.

Solution by Lawrence A. Ringenberg, Eastern Illinois State College. Let $C(n,p)$ denote the coefficient of the $(p+1)$st term in the binomial expansion of $(a+b)^n$. Simplifying the proportion,

$$C(n, p+q) : C(n, p+q+1) = (q+1) : (q+2),$$

we get the equation,

$$n(q+1) = p(2q+3) + 2(q+1)^2 \tag{1}$$

Setting $q = 0, 1, 2,$ in (1) we get an inconsistent system of equations. The values $q = 0$, 1 yield a consistent system and the desired values, $n = 14$, $p = 4$. Therefore the largest value of y is 3; the coefficients of the 5th, 6th, 7th terms in the expansion of $(a+b)^{14}$ are 1001, 2002, 3003.

Also solved by Maïmouna Edy, Hull P. Q. Canada, E. S. Keeping, *University of Alberta; M.S.Klamkin, Polytechnic Institute of Brooklyn, E.P.Starke, Rutgers University and the proposer.*

An Equilateral Triangle

237. [May 1955] *Proposed by M.N.Gopalan, Maharaja's College, Mysore, South India.*

Let p_1, p_2, p_3 be the altitudes and q_1, q_2, q_3 the medians of a triangle ABC. Prove:

(1) If $p_1p_2 + p_2p_3 + p_3p_1 = q_1q_2 + q_2q_3 + q_3q_1$ the triangle is equilateral.
(2) If $R(\tan A + \tan B + \tan C) = 2s$ where R is the circumradius and s is the semi-perimater then ABC is equilateral.
(3) If $p_1 + p_2 + p_3 = 9r$, where r is the inradius, then ABC is equilateral.

Solution by Maïmouna Edy, Hull P. Q. Canada.

(1) In any triangle we have $p_i \leq q_i$ $(i = 1, 2, 3)$ whence $p_1p_2 + p_2p_3 + p_3p_1 \leq q_1q_2 + q_2q_3 + q_3q_1$, the equality occurring if and only if $p_i = q_i$ for all i, i.e. if and only if the triangle is equilateral.

(2). *Notations.* We denote by A_i $(i = 1, 2, 3)$ the angles of the triangle, by a_i its sides and put $s_i = \sin A_i$, $t_i = \tan A_i$, finally $S = s_1 + s_2 + s_3$, $T = t_1 + t_2 + t_3$.

From $a_i = 2R\, s_i$ the given condition is seen equivalent to:

$$t_1 + t_2 + t_3 = 2(s_1 + s_2 + s_3) \text{ i.e.}$$

$$T = 2S.$$

Since in any triangle $t_1 + t_2 + t_3 = t_1t_2t_3$ (as is immediately seen by expanding the left hand side of $\tan(A_1 + A_2 + A_3) = 0$, the above condition also reads equivalently $t_1t_2t_3 = 2(s_1 + s_2 + s_3)$. In the latter relation the right hand side is positive, therefore so is the left hand side and hence all the angles of the triangle are acute. We now assert:

a) For all triangles $2S \leq 3\sqrt{3}$, equality occurring if an only if the triangle is equilateral.

b) For all acute angled triangles $T \geq 3\sqrt{3}$, equality occuring if and only if the triangle is equilateral.

The required result follows from a) and b) since $3\sqrt{3} \leq T = 2S \leq 3\sqrt{3}$ implies $T = 3\sqrt{3}$ and $2S = 3\sqrt{3}$, hence in virtue of a) or b) equality of sides.

Proof of a):

We have

$s_1 + s_2 = 2 \sin (A_1 + A_2)/2 \cos (A_1 - A_2)/2 = 2 \cos A_3/2 \cos (A_1 - A_2)/2$

If $A_1 \neq A_2$, that is, $\cos (A_1 - A_2)/2 < 1$ we can increase S by taking new angles $A_1' = A_2' = (A_1 + A_2)/2$, $A_3' = A_3$. So S is maximum for $A_1 = A_2 = A_3$ and then clearly $2S = 3\sqrt{3}$.

[Assertion a) amounts to the following: Of all triangles inscribed in a given circle the equilateral triangle has the largest perimeter.]

Proof of b):

We assume all the angles acute, hence $t_i > 0$ $(i = 1, 2, 3)$. The standard theorem on arithmetic and geometric means insures:

$$t_1 t_2 t_3 \leq [(t_1 + t_2 + t_3)/3]^3$$

with equality if and only if $t_1 = t_2 = t_3$.

But $t_1 t_2 t_3 = t_1 + t_2 + t_3 = T$, so the above inequality reads: $T \leq (T/3)^3$ i.e. $T^2 \geq 27$ or $T \geq 3\sqrt{3}$ with equality if and only if $t_1 = t_2 = t_3$, that is, if and only if the triangle is equilateral. This completes the proof.

(3). Let Δ denote the area of the triangle. In any triangle we have: $a_1 p_1 = a_2 p_2 = a_3 p_3 = 2\Delta = (a_1 + a_2 + a_3)(1/a_1 + 1/a_2 + 1/a_3)r$. Hence

$p_1 + p_2 + p_3 = r(a_1 + a_2 + a_3)(1/a_1 + 1/a_2 + 1/a_3) = r \; f(a_1, a_2, a_3)$

We shall have done if we can show that for any positive a_i $(i = 1,2,3)$ $f(a_1, a_2, a_3) \geq 9$, equality occurring if and only if $a_1 = a_2 = a_3$.

But the theorem on arithmetic and geometric means yields:

$$1/3 \; (a_1 + a_2 + a_3) \geq (a_1 a_2 a_3)^{1/3}$$

$$1/3 \; (1/a_1 + 1/a_2 + 1/a_3) \geq (1/a_1 a_2 a_3)^{1/3}$$

Multiplying member by member the latter relations we do get: $f(a_1, a_2, a_3) \geq 9$, equality occurring if and only if all the a_i are equal.

Also solved by Lawrence A. Ringenberg, Eastern Illinois State College, and the proposer.

Square Integers

225. [January 1955] *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Find an equality concerning squares of integers in which appear twelve consecutive squares and no others.

Solution by E. P. Starke, Rutgers University. Since $1^2 + 2^2 + \ldots + 12^2 = 650$ a solution is obtained by merely picking out squares whose

sum is 325, thus: (1) $9^2 + 10^2 + 12^2 = 1^2 + 2^2 + \ldots + 8^2 + 11^2$,
or we can write relations such as

$$(3^2 + 4^2) = 1^2 + 2^2 + 5^2 + 6^2 + \ldots + 12^2,$$

or if we wish to use the Pythagorean relation, the three "sides" should be $a^2 + b^2$, $a^2 - b^2$, $2ab$, whose sum is $2a(a + b) = 650$. So, $a = 13$, $b = 12$ give $25^2 + 312^2 = 313^2$, and we can write

$$(5^2)^2 + (1^2 + 3^2 + 6^2 + 8^2 + 9^2 + 11^2)^2 = (2^2 + 4^2 + 7^2 + 10^2 + 12^2)^2.$$

We can get a more general result than (1) by taking as the twelve consecutive squares:

(2) a^2, $(a + 1)^2$, ..., $(a + 11)^2$ whose sum is $12a^2 + 132a + 506$.

Six of the numbers (2), say a_1, a_2, ..., a_6, will have the same sum as the remaining six if we choose them such that

$$(3) \qquad \sum_{i=1}^{6} a_i = \sum_{i=7}^{12} a_i, \quad \sum_{i=1}^{6} a_i^2 = \sum_{i=7}^{12} a_i^2,$$

it takes but little manipulation to find such a set and to conclude that, for all a,

$$(a + 1)^2 + (a + 3)^2 + (a + 4)^2 + (a + 5)^2 + (a + 9)^2 + (a + 11)^2$$
$$= a^2 + (a + 2)^2 + (a + 6)^2 + (a + 7)^2 + (a + 8)^2 + (a + 10)^2.$$

Equations (3) are related to the Tarry-Escott problem and may easily be generalized. See Dickson, *History of the Theory of Numbers*, II, Chapter 24, and the article by Dorwart and Brown, *American Mathematical Monthly*, XLIV (1937), 613-633.

Also solved by Huseyin Demir, Zonguldak, Turkey: John M. Howell, Los Angeles City College; M. S. Klamkin, Brooklyn Polytechnic Institute; Leo Moser, University of Alberta; Chih-yi Wang, University of Minnesota and the proposer.

Moser noted that if the number twelve in the proposal is replaced by any odd number exceeding one, we have one of the equalities suggested by the following:

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$
$$\vdots$$

Stability of Solutions

238. [May 1955] *Proposed by Chih-yi Wang, University of Minnesota.*

Solve the differential equation:

$\frac{dx}{dt} = ax^3 + bx + c$, $x = x_0 \neq 0$ when $t = 0$ where a, b and c are real constants. Discuss the values of a, b and c which will give a stable solution, that is, in the solution, x remains bounded as t becomes infinite.

Solution by the proposer. Several special cases are very easy to solve.

Case I: $a = b = c = 0$, then $x = x_0$ (stable)

Case II: $a = b = 0$, then $x = ct + x_0$ (unstable for any real c, $c \neq 0$)

Case III: $a = c = 0$, then $x = x_0 e^{bt}$ (stable if $b < 0$)

Case IV: $b = c = 0$, then $x^2 = x_0^2/(1 - 2ax_0^2 t)$, (stable if $a < 0$)

Case V: $c = 0$, then

$$x^2 = \frac{bx_0^2 e^{2bt}}{ax_0^2(1 - e^{2bt}) + b} \quad \text{(stable if } b < 0 \text{ or } b/a < 0.)$$

Case VI: $a = 0$ then $x = (x_0 + c/b)\, e^{bt} - c/b$ (stable if $b < 0$)

Case VII: $b = 0$ then

$$\frac{(x+\alpha)\sqrt{x_0^2 - \alpha x_0 + \alpha^2}}{(x_0+\alpha)\sqrt{x^2 - \alpha x + \alpha^2}} = e^{3a\alpha^2 t}\, exp\, \{\sqrt{3}(\text{Arc tan}\, \frac{2x_0 - \alpha}{\sqrt{3}} - \text{Arc Tan}\, \frac{2x - \alpha}{3})\}$$

where $\alpha = \sqrt[3]{c/a}$. (stable)

General case: Let $b/a = p$, $c/a = q$, then the given differential equation can be written as

$$\text{(1)} \qquad \frac{dx}{x^3 + px + q} = adt.$$

Case (Gi). $\Delta = -4p^3 - 27q^2 > 0$. We have three distinct roots, say, r_1, r_2, r_3. Resolve the left hand side of (1) into partial fractions, we must have the following form

$$\frac{A}{x - r_1} + \frac{B}{x - r_2} + \frac{C}{x - r_3} \equiv \frac{1}{x^3 + px + q}$$

Since $A + B + C = 0$, A, B, and C cannot have the same sign. Then the unique solution of (1) is of the form, say,

$$\frac{(x - r_1)^{A'}(x_0 - r_2)^{B'}(x_0 - r_3)^{C'}}{(x_0 - r_1)^{A'}(x - r_2)^{B'}(x - r_3)^{B'}} = e^{at} \tag{2}$$

or its equivalent form, where A', B', C' are all positive and are assumed to be the solution of (2). It is obvious that the solution is stable for all real a.

Case (Gii). $\Delta < 0$. We have a single real root and two conjugate imaginary roots, say, r, $\beta + i\gamma$, $\beta - i\gamma$. Then the unique solution of (1) is

$$\frac{(x - r)\sqrt{(x_0 - \beta)^2 + \gamma^2}}{(x_0 - r)\sqrt{(x + \beta)^2 + \gamma^2}} =$$

$$\exp\{[(\beta - r)^2 + \gamma^2]\, at\} \exp\{\frac{r - \beta}{\gamma}[\text{Arc tan}\,\frac{x - \beta}{\gamma} - \text{Arc tan}\,\frac{x_0 - \beta}{\gamma}]\}$$

which is stable for all real a.

Case (Giii). $\Delta = 0$. We have two equal real roots and a distinct real root, say, r, r, r_1. The unique solution of (1) is

$$\frac{(x - r_1)(x_0 - r)}{(x - r)(x_0 - r_1)} = \exp\{(r - r_1)^2 at\} \exp\{(r - r_1)[\frac{1}{x - r} - \frac{1}{x_0 - r}]\}$$

which is stable for all real a.

Path Of A Moon

239. [May 1955] *Proposed by Norman Anning, Alhambra, California.*

In a certain solar system a planet has a moon. The position of the moon with respect to the sun of the system is an epicycle.

Solution by E. S. Keeping, University of Alberta. The curvature of the path is given by

$$\frac{1}{\rho} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

where $x' = dx/dt$, $x'' = d^2x/dt^2$, etc.

Substituting the given values of x and y, we find:

$$\frac{1}{\rho} = \frac{a^2 + n^3b^2 + n(n + 1)ab \cos (n - 1)t}{[a^2 + n^2b^2 + 2nab \cos(n - 1)t]^{3/2}}$$

This is of constant sign (for a, b, n all positive) if $a^2 + b^2n^3 > n\ ab(n+1)$ i.e. if $(a - nb)(a - n^2b) > 0$.

Since $a > b$, the condition is $n^2 < a/b$ or $n > a/b$. For the data supplied, $n^2 < a/b$, so that the curvature is everywhere positive.

The distance of any point on the path from the origin is given by $r^2 = a^2 + b^2 + 2ab \cos (n-1)t$, so that the maximum distance is $a + b$ and the minimum $a - b$. At each maximum the curvature is $(a + n^2b)/(a + nb)^2$, and at each minimum it is $(a - n^2/b)/(a - nb)^2$. The smallest n for which the curvature is zero is therefore $n = \sqrt{a/b} = 19.7$ from the given data.

Also solved by Maïmouna Edy, Hull, P. Q. Canada. Edy pointed out that in addition to the above solution, if $(a^2 + b^2n)/(ab + abn) = (a^2 + b^2n^3)/(abn + abn^2) = 0$ we have $n = a/b$. For this exceptional n, though larger than $\sqrt{a/b}$, the curve shows no inflections. In this case $n = 387.5$.

Triple Vector Product

240. [May 1955] *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Determine the value of

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B)$$

without expanding any of the triple vector products.

Solution by Samuel Skolnik, Los Angeles City College. If A, B and C are coplanar vectors or if one of them is a null vector, the solution is trivial.

Assume that A, B and C are non-coplanar and let $P = A\times(B\times C) + B\times(C\times A) + C\times(A\times B)$. Then

$$A \cdot P = A \cdot A \times (B \times C) + A \cdot B \times (C \times A) + A \cdot C \times (A \times B)$$

$$= 0 + (A \times B) \cdot (C \times A) + (A \times C) \cdot (A \times B)$$

$$= (A \times B) \cdot (C \times A) - (C \times A) \cdot (A \times B) = 0$$

Similarly $B \cdot P = 0$ and $C \cdot P = 0$. Since P could not be perpendicular to three non-coplanar vectors A, B and C it follows that $P = 0$.

Also solved by Huseyin Demir, Zonguldak, Turkey; Maïmouna Edy, Hull P. Q., Canada; T. F. Mulcrone, St. Charles College, Louisiana; Chih-yi Wang, University of Minnesota and the proposer.

Rod In A Corridor

241. [May 1955] *Proposed by Leon Bankoff, Los Angeles California.*

A thin rod of length L is the longest that can be moved horizontally from one corridor into another at right angles to the first. When the rod touches the inner corner and the two outer walls of the corridors, its inclination to the two walls is θ and $(90° - \theta)$. Given L and θ find a and b, the widths of the two corridors.

Solution by Sam Kravitz, East Cleveland, Ohio. Let rod AOB touch the outer walls at A and B, and the inner corner at O. Point C is the corner of the outer walls. From O drop perpendiculars a meeting BC at M and b meeting AC at N. Angle ABC equals angle AON equals θ.

Now $L = b \sec \theta + a \csc \theta$

$$\frac{dL}{d\theta} = b \sec\theta \tan\theta - a \csc\theta \operatorname{ctn}\theta = 0 \text{ for an extreme value.}$$

Thus
$$\frac{b}{a} = \frac{\csc\theta \operatorname{ctn}\theta}{\sec\theta \tan\theta} = \operatorname{ctn}^3\theta \text{ or } b = a \operatorname{ctn}^3\theta.$$

Substituting this value in the equation for L we have

$$a = \frac{L}{\operatorname{ctn}^3\theta \sec\theta + \csc\theta} = L \sin^3\theta \text{ and } b = L \cos^3\theta.$$

Also solved by Huseyin Demir, Zonguldak, Turkey; Maimouna Edy, Hull, P. Q., Canada; A. L. Epstein, Cambridge Research Center, Massachusetts; M. S. Klamkin, Polytechnic Institute of Brooklyn; Louis S. Mann, Los Angeles, California; Lawrence A. Ringenberg, Eastern Illinois State College; N. Shklov, University of Saskatchewan; Chih-yi Wang, University of Minnesota and the proposer.

Homothetic Triangles

242. [May 1955] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let A', B', C' be the points dividing the sides of triangle ABC in the ratio k, and let A'', B'', C'' be the points dividing the sides of triangle $A'B'C'$ in the ratio $1/k$. Prove that the triangle $A''B''C''$ is homothetic with the original triangle ABC.

Solution by P. W. Allen Raine, Newport News High School, Newport News, Virginia. Let A, B, C, A', B', C', A'', B'', C'' represent the vector coordinates of the respective points and k, a scalar quantity. Thus

$$A' = \frac{kB + C}{k+1}, \quad B' = \frac{kC + A}{k+1}, \quad C' = \frac{kA + B}{k+1}$$

and

$$A'' = \frac{B' + kC'}{1+k} = \frac{kC + A + k^2A + kB}{(1+k)^2},$$

$$B'' = \frac{B' + kA'}{1+k} = \frac{kA + B + k^2B + kC}{(1+k)^2},$$

$$C'' = \frac{A' + kB'}{1 + k} = \frac{kB + C + k^2C + kA}{(1 + k)^2}$$

Now we can easily show that

$$A'' - B'' = \frac{1 - k + k^2}{(1 + k)^2}(A - B),$$

$$B'' - C'' = \frac{1 - k + k^2}{(1 + k)^2}(B - C),$$

$$C'' - A'' = \frac{1 - k + k^2}{(1 + k)^2}(C - A)$$

which tells us that the sides of the two triangles are parallel and hence the triangles are homothetic, the homothetic ratio being

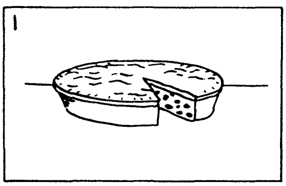
$$\frac{1 - k + k^2}{(1 + k)^2}$$

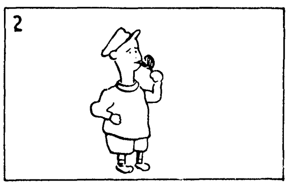
Also solved by Maimouna Edy, Hull, P. Q., Canada; M. S. Klamkin, Polytechnic Institute of Brooklyn; Chih-yi Wang, University of Minnesota and the proposer.

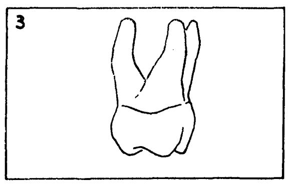
Mathematical Ideography

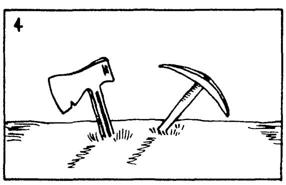
222. [January 1955] *Proposed by C. W. Trigg, Los Angeles City College and Leon Bankoff, Los Angeles, California.*

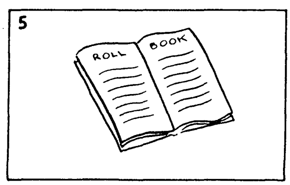
Translate each of the following sketches into a mathematical term.

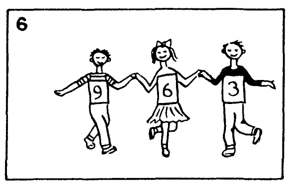




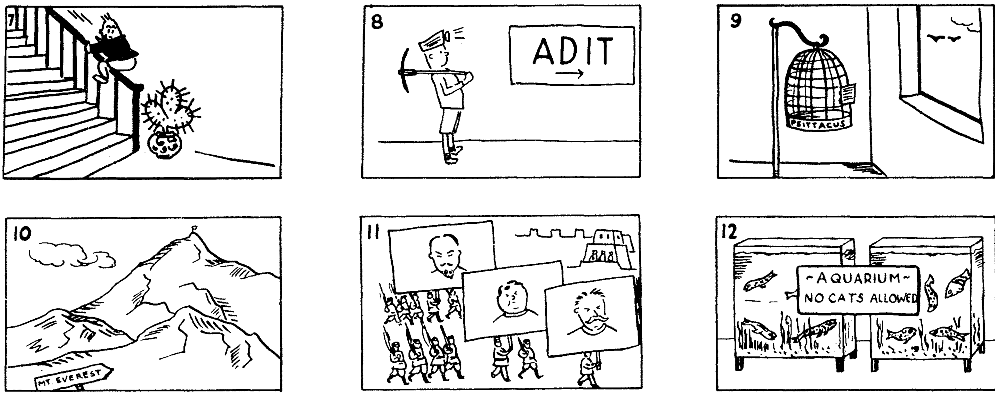








Solution by the proposers (1) Pie = π, (2) He licks = Helix, (3) Triple root, (4) Oblique axes, (5) Null class, (6) Amicable numbers,



(7) Method of descent, (8) Miner = minor, (9) Poly gone = Polygon, (10) Upper limit, (11) Superposed radicals, (12) Nary a cat = catenary.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q. 159. Along a straight path are a number of houses not necessarily equally spaced. In each house are a number of men, not necessarily the same number of men in each house. If all of these men wish to get together for a meeting at which house shall they meet so that the total amount of walking shall be a minimum? [*Submitted by Julian H. Braun*]

Q. 160. Find the sum of $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 + 144$. [*Submitted by M. S. Klamkin*]

Q. 161. Quickie 151 has a pup! A man is waiting to put through a person to person call to Sinkiang. He begins to write the number 0.12345... which has n in the nth place of decimals. Being a tidy doodler, he attends promptly to all "carrying figures". Show that he may get his message through before needing to write the digit 8. [*Submitted by Norman Anning.*]

Q. 162. (From the 1953 Putman Competition) Six points are in general position in space, no three in a line and no four in a plane. The fifteen line segments joining them in pairs are drawn and then painted, some segments red and some blue. Prove that some triangle has all of its sides the same color. [*Submitted by M. S. Klamkin*]

Q. 163. Prove that $1 + 1/2 + 1/3 + \ldots + 1/n$ is never an integer for $n > 1$. [*Submitted by M. S. Klamkin*]

Q. 164. If $\frac{F(m) + F(n)}{2} \geq F\left(\frac{m+n}{2}\right)$ is true for all real m and n, prove that $\frac{F^{-1}(m) + F^{-1}(n)}{2} \leq F^{-1}\left(\frac{m+n}{2}\right)$ in a domain where the inverse function $F^{-1}(x)$ exists. [*Submitted by M. S. Klamkin*]

ANSWERS

A. 159. They should meet in the house where the middle man is. If there are two middle men and they happen to be in different houses, either house will do. Comment: Suppose the path is a simply connected closed curve perhaps surrounding a lake. As there is no middle man, is there a simple answer in this case?

A. 160.

$$\text{Let } S = \sum_{n=1}^{12} U_n = \sum_{n=1}^{12} (U_{n+2} - U_{n+1})$$

$$= U_{14} - U_2$$

$$\text{So } S = 2(144) + 89 - 1 = 376$$

A. 161. The $\lim_{k\to\infty} \sum_1^k \frac{n}{10^n} = \frac{10}{81} = 0.\underline{123456790}\ldots$ a periodic decimal not containing the integer 8.

A. 162. There are five segments emanating from any point. Three of these must be of the same color say red. No matter how we connect the ends of these three segments we get at least one triangle of the same color.

A. 163. Multiply the sum by one half the least common multiple. Then there will be exactly one term equal to 1/2 and the remaining terms will be integers since there can be only one term which contains the highest power of 2 in the sequence 1, 2, 3 ... n.

A. 164. Geometrically this is equivalent to proving that if a function is convex its inverse is never convex. Plot the curves $y = F(x)$ and $y = F^{-1}(x)$. These curves are mirror images in the line $y = x$. Thus the proof follows by symmetry.

(OUR CONTRIBUTORS *continued)*

Articles by the following authors appeared in the Nov.-Dec. issue.

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Rose Lariviere is a native of Montreal, Canada and a graduate of McGill University. After doing teaching and research in physics she obtained her M.A. in mathematics from the University of Illinois, and has been teaching at the Chicago Undergraduate Division of this University since 1946.

L. E. Diamond was born in Grand Rapids, Michigan, in 1887. He was in business in New York, Chicago and Italy until 1940. His interest in mathematics was aroused in 1938 by an inspiring teacher, Dr. Harry Everett of the University of Chicago. He holds the degrees of B.S., University of Chicago (1941) and M.S., University of Oklahoma (1942).

William B. Orange Competition

We have been requested to say that the William B. Orange Competitive examinations are held by the Mathematics Department of the Los Angeles City College in memory of Prof. Orange. (see MATHEMATICS MAGAZINE Vol. 29, No. 2)

Editor